SCATTERING FOR SCHRÖDINGER OPERATORS WITH CONICAL DECAY

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Abstract. We study the scattering properties of Schrödinger operators with potentials that have short-range decay along a collection of rays in $\mathbb{R}^d$. This generalizes the classical setting of short-range scattering in which the potential is assumed to decay along *all* rays. For these operators, we give a microlocal characterization of the scattering states in terms of the dynamics and a corresponding description of their complement. This shows that any state decomposes into an asymptotically free piece and a piece that may interact with the potential for long times. We also show that in certain cases these characterizations can be purely spatial.

Contents

1. Introduction

In this paper, we study the scattering properties of Schrödinger operators with potentials that have short-range decay along a collection of rays in $\mathbb{R}^d$. This generalizes the classical setting of short-range scattering in which the potential is assumed to decay along *all* rays. By now, the scattering theory of short-range potentials is classical, but much less is known about the anisotropic setting in which the behavior of the potential at infinity depends on the direction. Here, we study how purely geometric constraints on the potential affect scattering. Namely, we consider real-valued potentials that are concentrated near a subset of $\mathbb{R}^d$ with complement containing rays to infinity. We show that microlocally the scattering states are precisely those that concentrate along these rays in phase space. States in the complement of the scattering states, called the "interacting subspace" below, thus must avoid these rays in a suitable sense. We remark that unlike in the short-range theory, these interacting states need not be pure point so that they may form interesting examples of continuous states satisfying some geometric confinement condition - see the discussion of surface states in [3] for several examples. Besides boundedness and the aforementioned geometric condition, we make no other assumptions on the potential so that in particular it may be very rough or have wild behavior at infinity in some directions.

Let us now recall the classical picture of short-range scattering in order to situate our result. We consider a self-adjoint Schrödinger operator of the form

$$H = H_0 + V$$

on $\mathcal{H} = L^2(\mathbb{R}^d)$ where $H_0 = -\frac{1}{2}\Delta$ and V is a real-valued bounded multiplication operator. There are a variety of decay conditions one can impose on V in order to consider it short-range (see [1] and the references therein), but we focus on the *Enss condition*

$$\|\chi_{B_r^c} V\|_{\text{op}} \in L^1([0,\infty), dr) \tag{1.1}$$

where χ is the indicator function of a subset of $\mathbb{R}^d$ and B_r is the ball of radius r in $\mathbb{R}^d$. This condition was originally posited in [9], in which it was proven that the wave operators

$$\Omega^{\pm} = \operatorname*{s-lim}_{t\to\mp\infty} e^{itH} e^{-itH_0}$$

whose ranges consist of the scattering states, exist on all of $\mathcal{H}$, and are asymptotically complete. This means that

$$\mathcal{H} = \operatorname{Ran}(\Omega^{\pm}) \oplus \mathcal{H}_{\text{pp}}(H)$$

or equivalently

$$\mathcal{H}_{\text{c}}(H) = \operatorname{Ran}(\Omega^{\pm}) \tag{1.2}$$

The proof of this result due to Enss [9], as well as its refinement by Davies [5], relies on studying the phase space localization of a state as it evolves under H. Ultimately, the scattering states are characterized dynamically via the celebrated RAGE theorem [2, 14].

Motivated by this classical theory, in [3] we studied the scattering properties of potentials assumed to decay only in some coordinate directions. Formally, if S_r is the set of points of a distance less than r from some subspace of $\mathbb{R}^d$ then we studied potentials satisfying the *subspace Enss condition*

$$\|\chi_{S_r^c} V\|_{\text{op}} \in L^1([0,\infty), dr)$$

In this setting, we showed that Ω^- exists for all $\psi \in \mathcal{H}$ and that the orthogonal complement of its range is given by the set of surface states

$$\mathcal{H}_{\text{sur}} = \{\psi \in \mathcal{H} \mid \forall v > 0, \lim_{t\to\infty} \|\chi_{S_{vt}^c} e^{-itH}\psi\| = 0\}$$

so that

$$\mathcal{H} = \operatorname{Ran}(\Omega^-) \oplus \mathcal{H}_{\text{sur}}$$

Thus, even though asymptotic completeness in the sense of (1.2) does not generally hold in this setting, we were still able to provide a dynamical characterization of the non-scattering states, albeit not a spectral one. That work proceeds via the phase space scattering methodology of Enss and makes heavy use of the subspace structure of the potential and the tensor product structure of $\mathbb{R}^d$. However, one expects to have a robust theory of scattering for a larger class of geometrically constrained potentials. Indeed, a potential that admits asymptotically free trajectories should produce scattering states, at least intuitively. Classically, one expects that a particle moving under the influence of a potential may escape along some ray so long as the strength of the potential attenuates fast enough along that ray. Here, we study the quantum analog of this phenomenon: we let V decay inside a (possibly infinite) collection of convex cones. For a ray in the interior of a cone, the distance to the boundary of the cone increases along the ray. Thus, the use of cones enforces that the effect of the potential must decrease along any classically free trajectory, while the convexity is a technical convenience.

In the anisotropic setting and others with multiple channels of scattering, the main challenge is that not all continuous states will have their dynamics governed by the RAGE theorem, which is the starting point of the Enss method. For time-dependent problems, some authors have circumvented this difficulty by defining the space of scattering states via their dynamical properties and then, when possible, showing that these states are precisely those in the range of a wave operator. Though the class of potentials considered (time-dependent and spatially decaying in all directions) are quite different, the works of Kitada and Yajima [12] and Yafaev [20] are influential to our methodology. In the former, a *microlocal* characterization of the set of scattering states is developed while in the latter the author defines the scattering states in analogy to the RAGE theorem as those that leave any compact set, in the appropriate sense.

Before specifying the class of V more precisely, we mention some lines of inquiry that are connected to our results. Previously, Yafaev et al [8, 18, 21] made a deep study of certain types of anisotropic potentials, as was recalled in Section 6.3 of [3]. In this series of papers, the authors considered potentials that decay short-range in some coordinate directions, but slower in others. They constructed many examples of potentials in this class for which asymptotic completeness does not hold and proved the existence of modified wave operators for the states in the complement of the range of the standard wave operator. Thus, they show that multiple channels of scattering may appear in the presence of anisotropy if the potential is chosen carefully. In contrast, the present paper is concerned with a much more general class of anisotropic potentials. Our main theorems provide constraints on how asymptotic completeness may fail by showing how states in non-free channels must behave micro-locally, but we do not show that any specific potential actually exhibits this failure (for such examples, see Section 6 of [3] or [19]). And so, the potentials considered in [18, 19] fall within the purview of our results and provide important demonstrations of the phenomenon we wish to study. Furthermore, previous work has concentrated on anisotropy with respect to orthogonal subspaces of $\mathbb{R}^d$, while our goal is to study potentials that have arbitrary decay along each ray towards infinity.

Less closely related, but still relevant is the recent progress in understanding how the geometry of the potential affects the spectrum of H, especially in the study of geometrically-induced bounds states. One representative example is [10] in which a condition is given for the existence of bound states due to singular potentials supported on certain curves in $\mathbb{R}^2$. Where our theorem applies to these settings, such states will appear in the interacting subspace $\mathcal{H}_{\text{int}}$ (to be defined below). Beyond this, one may place the geometry in the underlying space instead of in the potential by studying short-range scattering on a manifold as in [11].

There are also many directions for further research suggested by our results. First, it is quite natural to ask whether one can give some sort of spectral criteria for the presence of interacting states since at present our characterization is purely dynamical. Furthermore, under additional assumptions on the potential, the dynamics or spectral properties of the interacting states are themselves worth investigating. Perhaps most importantly, if V is partially (quasi-)periodic, one can ask whether the potential acts as a waveguide in the sense that it produces ac states that propagate in its vicinity (see [3] for a partial answer to this question). Given the presence of such ac states, one may ask many questions about their dynamics, for instance, if they exhibit anisotropic ballistic transport and whether they satisfy anisotropic dispersive estimates. In an unrelated direction, it would be interesting to construct potentials with conical decay that produce singular continuous states and to study their dynamics. More physically, one could study conical scattering with periodic background, i.e by replacing H_0 with $H_0 + W$ where W is periodic. This would model a crystal with a conical defect. The Enss method has been employed for periodic background [16], but the generalization to conical decay presents some technical challenges.

2. Model

As mentioned above, we consider a self-adjoint operator H on $\mathcal{H} = L^2(\mathbb{R}^d)$ of the form

$$H = H_0 + V \tag{2.1}$$

where $H_0 = -\frac{1}{2}\Delta$ and V is a real-valued bounded potential that decays inside a collection of convex cones.

To be more precise, let us first fix some notation. For any $x \in \mathbb{R}^d, \vec{v} \in \mathbb{S}^{d-1}$, and $\gamma \in (0, \pi)$ let

$$\mathcal{C}_{x,\gamma,\vec{v}} = \{y \in \mathbb{R}^d \mid \langle (y - x), \vec{v}\rangle > \cos(\gamma)\|y - x\|\}$$

be the open cone with vertex x in the $\vec{v}$ direction with aperture 2γ. Since we will often work with cones with vertex at the origin, we let $\mathcal{C}_{\vec{v},\gamma}$ denote $\mathcal{C}_{0,\vec{v},\gamma}$. Unless otherwise specified, all cones will be assumed to be convex or equivalently have $\gamma \leq \frac{\pi}{2}$. Furthermore, for any cone $\mathcal{C}$, let $A_r(\mathcal{C})$ be the set of points a distance greater than $r > 0$ from $\mathcal{C}^c$:

$$A_r(\mathcal{C}) = \{y \in \mathbb{R}^d \mid d(y, \mathcal{C}^c) > r\}$$

See Figure 1. We will use the shorthand $A_r^c(\mathcal{C}) = [A_r(\mathcal{C})]^c$.

For some collection of convex cones $\{\mathcal{C}_i\}_{i\in\mathcal{I}}$, let

$$\mathcal{A}_r = \bigcup_{i\in\mathcal{I}} A_r(\mathcal{C}_i)$$

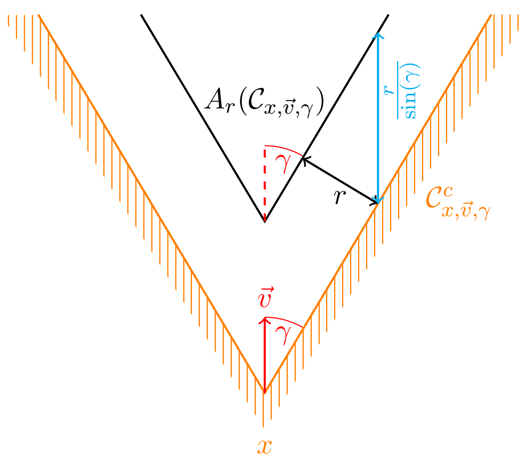

FIGURE 1. Illustration of $\mathcal{C}_{x,\vec{v},\gamma}$ and $A_r(\mathcal{C}_{x,\vec{v},\gamma})$ for $d = 2$: in orange we have the complement of $\mathcal{C}_{x,\vec{v},\gamma}$, which is where the potential is concentrated. In black we have the set $A_r(\mathcal{C}_{x,\vec{v},\gamma})$, in red we indicate $\vec{v}$ and γ.

We assume that V satisfies the following *generalized Enss condition* with respect to $\{\mathcal{C}_i\}_{i\in\mathcal{I}}$:

$$\|\chi_{\mathcal{A}_r} V\|_{\mathrm{op}} \in L^1([0,\infty), dr)$$

which should be compared to (1.1). Note that $\mathcal{A}_r$ depends implicitly on the collection $\{\mathcal{C}_i\}_{i\in\mathcal{I}}$ and therefore this condition depends only on the geometry of V.

We will study the scattering properties of H via the (positive time) wave operator Ω^-, which we simply write as Ω. Our results may be easily reformulated for Ω^+, but we focus our attention on the limit $t \to \infty$. Before stating a precise theorem (see Section 3), let us give a few examples of the geometries we plan to consider and explain how they may be described via a union of convex cones.

Example 2.1 (Single cone)**.** It is already interesting to consider V which decays in some convex cone $\mathcal{C}$, that is, $\{\mathcal{C}_i\}_{i\in\mathcal{I}}$ consists of a single cone. For such potentials, we will show that $\mathrm{Ran}(\Omega)$ consists of states which evolve into $\mathcal{C}$ with momenta lying in $\mathcal{C}$ whereas $\mathrm{Ran}(\Omega)^\perp$ consists of states which may interact with V for arbitrarily long times. These characterizations are microlocal in the sense that they depend on the position and momentum localization of a state.

Example 2.2 (Non-convex cone)**.** The use of convex cones (that is, $\gamma \leq \frac{\pi}{2}$) is merely a technical convenience: a cone with $\gamma > \frac{\pi}{2}$ can be described as an intersection of a collection of half-spaces, and therefore falls within the purview of the analysis below. To see this, say $\mathcal{C} = \mathcal{C}_{0,\vec{v},\gamma}$ for $\gamma > \frac{\pi}{2}$ and define

$$\mathscr{S}_{\vec{v}} = \{\vec{w} \in \mathbb{S}^{d-1} \mid \langle \vec{v}, \vec{w}\rangle = \cos(\gamma - \frac{\pi}{2})\}$$

the conic envelope. Then one may check that

$$\mathcal{C} = \bigcup_{\vec{w}\in\mathscr{S}_{\vec{v}}} \mathcal{C}_{0,\vec{w},\frac{\pi}{2}}$$

Example 2.3 (Short-range scattering)**.** The generalized Enss condition encompasses the classical Enss condition (1.1) for short-range potentials. Indeed, one may study short-range potentials in the present setting by writing

$$B_r^c = \bigcup_{\vec{v}\in\mathbb{S}^{d-1}} A_r(\mathcal{C}_{\vec{v},\frac{\pi}{2}})$$

which may be readily verified.

Example 2.4 (Subspace potentials)**.** In [3], we studied potentials that are supported near a subspace of $\mathbb{R}^d$, as explained in Section 1. Using the product structure of this geometry, we proved that Ω exists for all $\psi \in \mathcal{H}$ and gave a purely spatial characterization of $\mathrm{Ran}(\Omega)^\perp$. We will show that one may recover these results in the present setting since by similar considerations as in the above example, it is easy to see that

$$S_r^c = \bigcup_{\vec{v}\in\vec{0}_k\times\mathbb{S}^{d-k-1}} A_r(\mathcal{C}_{\vec{v},\frac{\pi}{2}})$$

where $\vec{0}_k = (0,\dots,0) \in \mathbb{R}^k$.

Example 2.5 (Broken subspace)**.** A variant of the above example is a "broken subspace," written here in $d = 2$ for simplicity: consider $\vec{v}_1, \vec{v}_2 \in \mathbb{S}^1$ and let $\vec{r}_1$ and $\vec{r}_2$ be the rays $\{t\vec{v}_1 \mid t \geq 0\}$ and $\{t\vec{v}_2 \mid t \geq 0\}$, respectively. Then consider V such that

$$\operatorname{supp} V \subset T_r := \{x \in \mathbb{R}^2 \mid d(x, \vec{r}_1 \cup \vec{r}_2) < r\}$$

We may accommodate such potentials by observing that

$$T_r = (\mathcal{C}_{r\vec{v}_*,\vec{v}_*,\gamma} \cup \mathcal{C}_{-r\vec{v}_*,-\vec{v}_*,\pi-\gamma})^c$$

where $\vec{v}_* = \frac{\vec{v}_1+\vec{v}_2}{\|\vec{v}_1+\vec{v}_2\|}$ and γ is half of the (non-obtuse) angle between $\vec{v}_1$ and $\vec{v}_2$, see Figure 2. In this example, one of the cones is convex while the other is not.

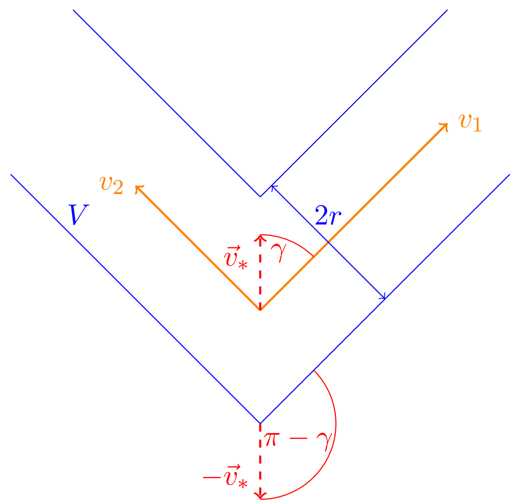

FIGURE 2. The geometry of the broken subspace: in orange, we have the vectors v_1, v_2, in red we have the vectors $\vec{v}_*$ and $-\vec{v}_*$, in blue we have the outline of T_r, which contains $\operatorname{supp} V$.

3. Definitions and Results

3.1. Notation and Conventions.

- We let $\mathcal{H}$ denote $L^2(\mathbb{R}^d)$ with norm denoted $\|\cdot\|$ and use the convention that its inner product $\langle\cdot,\cdot\rangle$ is anti-linear in the first argument and linear in the second.
- The symbols $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ will also be used for the norm and inner product on $\mathbb{R}^d$.
- $d(\cdot,\cdot)$ is used for the distance between points or subsets of $\mathbb{R}^d$.
- B_r will mean the ball of radius r centered at the origin in either $\mathbb{R}^d$ or $\mathcal{H}$ depending on context.
- For $A \subset \mathbb{R}^d$, A^c denotes its complement.
- χ_A will mean the indicator function of a set $A \subset \mathbb{R}^d$.
- $A \Subset B$ denotes that A is compactly contained in B.
- $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$, the Schwartz space.
- We use the following convention for the Fourier transform of $f \in \mathcal{H}$:

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = (2\pi)^{-\frac{d}{2}} \int\limits_{\mathbb{R}^d} f(x)e^{-ix\xi}\,dx$$

$$\mathcal{F}^{-1}(\hat{f})(x) = (2\pi)^{-\frac{d}{2}} \int\limits_{\mathbb{R}^d} \hat{f}(\xi)e^{ix\xi}\,d\xi$$

- For some cone $\mathcal{C}_{x,\vec{v},\gamma}$ and $r > 0$, we define $A_r(\mathcal{C}_{x,\vec{v},\gamma}) \subset \mathbb{R}^d$ to be the set of all points at a distance greater than r from $\mathcal{C}^c_{x,\vec{v},\gamma}$

$$A_r(\mathcal{C}_{x,\vec{v},\gamma}) = \{y \in \mathbb{R}^d \mid d(y, \mathcal{C}^c_{x,\vec{v},\gamma}) > r\}$$

  As explained below,

$$A_r(\mathcal{C}_{x,\vec{v},\gamma}) = \mathcal{C}_{x,\vec{v},\gamma} + \frac{r}{\sin(\gamma)}\vec{v}$$

which we will use to define A_r for $r \leq 0$. We will also use the shorthand

$$A_r^c(\mathcal{C}_{x,\vec{v},\gamma}) = [A_r(\mathcal{C}_{x,\vec{v},\gamma})]^c$$

- We will let

$$\mathcal{A}_r = \bigcup_{i \in \mathcal{I}} A_r(\mathcal{C}_i)$$

for $\{\mathcal{C}_i\}_{i \in \mathcal{I}}$ some collection of cones.
- For some cone $\mathcal{C}_{x,\vec{v},\gamma}$ and $k > 0$ we let

$$\mathcal{D}_k(\mathcal{C}_{x,\vec{v},\gamma}) = \{\psi \in \mathcal{S} \mid \operatorname{supp} \hat{\psi} \Subset A_k(\mathcal{C}_{\vec{v},\gamma})\}$$
$$\mathcal{D}(\mathcal{C}_{x,\vec{v},\gamma}) = \{\psi \in \mathcal{H} \mid \operatorname{supp} \hat{\psi} \subset \mathcal{C}_{\vec{v},\gamma}\}$$

- For the definition of $P_\delta(\cdot)$ see Appendix A.
- ψ_t will always denote the evolution of ψ under H at time t:

$$\psi_t = e^{-itH}\psi$$

- We will also use the following notation:

$$\Omega(t) = e^{itH}e^{-itH_0}$$
$$\Omega^*(t) = e^{itH_0}e^{-itH}$$

and

$$\Omega = \Omega^- = \operatorname*{s-lim}_{t \to +\infty} e^{itH}e^{-itH_0}$$

with domain $\mathcal{D}$ that will be described below.
- $\operatorname{Ran}(\Omega)$ will refer to the range of Ω on its natural domain $\mathcal{D}$.

3.2. **Definition of the scattering and interacting subspaces.** In order to give the aforementioned microlocal characterizations, we will need a suitable way to describe a state's localization in phase space. To this end, for every $\delta > 0$, we define a positive operator-valued measure (POVM), denoted P_δ, on the phase space $\mathbb{R}^d_x \times \mathbb{R}^d_p$, with the following properties:

(1) (Observable) $P_\delta(\mathbb{R}^{2d}) = \mathrm{id}$.
(2) (Momentum localization) Let $B \subset \mathbb{R}^d$ and $D \subset \mathbb{R}^d$ be Borel sets such that $d(B, D) > \delta$. Then for any $E \subset \mathbb{R}^d \times B$ Borel and $\psi \in \mathcal{H}$ such that $\operatorname{supp} \hat{\psi} \subset D$

$$P_\delta(E)\psi = 0$$

(3) (Approximate space localization) Let $A \subset \mathbb{R}^d$ and $D \subset \mathbb{R}^d$ be Borel sets such that $d(D, A) > 0$. Then for any $\ell > 0$ there exists some constant $C > 0$ depending only on η_δ so that for all $E \subset A \times \mathbb{R}^d$

$$\|P_\delta(E)\chi_D\|_{\mathrm{op}} < C[d(A, D)]^{-\ell}$$

(4) (Microlocal non-stationary phase estimate) Let $\mathfrak{C}_t(E) \subset \mathbb{R}^d$ denote the classically allowed region associated to $E \subset \mathbb{R}^{2d}$ at time t:

$$\mathfrak{C}_t(E) = \{x + tp \mid (x, p) \in E\}$$

Let $F \subset \mathbb{R}^d$ be Borel. For any $\ell > 0$ there exists $C > 0$ such that

$$\|\chi_F e^{-itH_0} P_\delta(E)\|_{\mathrm{op}} \leq Cd(|t|)^{-\ell}$$

for all t such that $d(t) := d(\mathfrak{C}_t(E), F) > \delta|t|$.

(5) (Spatial non-stationary phase estimate) Let $\{A_t\}_{t\geq 0}$ be a collection of Borel subsets of $\mathbb{R}^d$. Then for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ Schwartz such that $\operatorname{supp}\hat{\varphi} \Subset D$ Borel, $\ell > 0$, and $\varepsilon > 0$ there exists some constant $C(\psi, \ell, \varepsilon, \delta) > 0$ such that

$$\|P_\delta(A_t \times \mathbb{R}^d)e^{-itH_0}\varphi\| < Ct^{-\ell}$$

for all t such that $d(A_t, tD) > \varepsilon t$.

We refer the reader to [4] for the definition of a POVM and relegate the construction of a POVM satisfying the above properties to Appendix A.

To specify the domain of Ω, we let

$$\mathcal{D}(\mathcal{C}_{x,\vec{v},\gamma}) = \{\psi \in \mathcal{H} \mid \operatorname{supp}\hat{\psi} \subset \mathcal{C}_{\vec{v},\gamma}\}$$

In particular, $\mathcal{D}(\mathcal{C}_{x,\vec{v},\gamma})$ is independent of the vertex x. For some collection of cones $\{\mathcal{C}_i\}_{i\in\mathcal{I}}$, we let

$$\mathcal{D} = \overline{\bigcup_{i\in\mathcal{I}} \mathcal{D}(\mathcal{C}_i)}$$

For $n > 0$ and some cone $\mathcal{C}_{x,\vec{v},\gamma}$, we let the corresponding *outgoing* subset of phase space be the set of points with space coordinates in $A_n(\mathcal{C}_{x,\vec{v},\gamma})$ and momentum coordinates in $\mathcal{C}_{\vec{v},\gamma}$:

$$W_{n;\mathrm{out}}(\mathcal{C}_{x,\vec{v},\gamma}) = \{(y,p) \in \mathbb{R}^{2d} \mid y \in A_n(\mathcal{C}_{x,\vec{v},\gamma}) \text{ and } p \in \mathcal{C}_{\vec{v},\gamma}\}$$

and let the total outgoing subset be

$$\mathcal{W}_{n;\mathrm{out}} = \bigcup_{i\in\mathcal{I}} W_{n;\mathrm{out}}(\mathcal{C}_i)$$

We also define a variant of $W_{n;\mathrm{out}}(\mathcal{C})$ which is restricted away from 0 in the momentum variable:

$$W_{n,m;\mathrm{out}}(\mathcal{C}_{x,\vec{v},\gamma}) = \{(y,p) \in \mathbb{R}^{2d} \mid y \in A_n(\mathcal{C}_{x,\vec{v},\gamma}) \text{ and } p \in A_m(\mathcal{C}_{\vec{v},\gamma})\}$$

and its respective total set

$$\mathcal{W}_{n,m;\mathrm{out}} = \bigcup_{i\in\mathcal{I}} W_{n,m;\mathrm{out}}(\mathcal{C}_i)$$

See Figure 3.

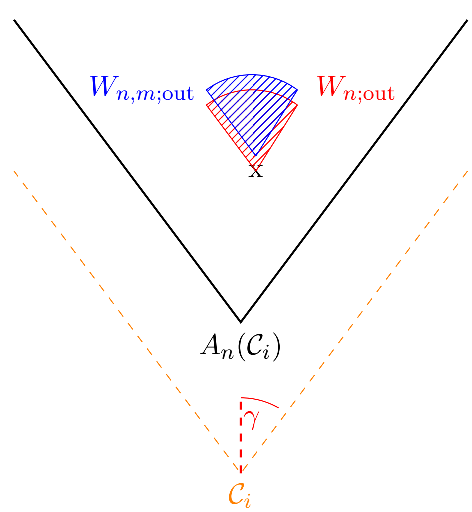

FIGURE 3. Illustration of the phase space sets $W_{n;\mathrm{out}}(\mathcal{C}_i)$ and $W_{n,m;\mathrm{out}}(\mathcal{C}_i)$: space coordinates are inside the black cone while momentum coordinates point inside the red/blue cone, respectively.

This allows us to define the scattering subspace

$$\mathcal{H}_{\rm scat} = \{\psi \in \mathcal{H} \mid \exists v, m, \delta_0 > 0 \text{ so that } \forall \delta \in (0, \delta_0) \lim_{t\to\infty} \|(P_\delta(\mathcal{W}_{vt,m;\rm out}) - \mathrm{Id})\psi_t\| = 0\}$$

which we will prove below is dense in $\mathrm{Ran}(\Omega) := \Omega(\mathcal{D})$.

Remark 3.1. The above characterization of $\mathrm{Ran}(\Omega)$ is similar to those given in [12] and [22] in the short-range setting.

We also define the interacting subspace

$$\mathcal{H}_{\rm int} = \{\psi \in \mathcal{H} \mid \forall v, m > 0, \exists \delta_0 > 0 \text{ so that } \forall \delta \in (0, \delta_0) \lim_{t\to\infty} \|P_\delta(\mathcal{W}_{vt,m;\rm out})\psi_t\| = 0\}$$

which consists of states that can interact with V for arbitrarily long times. We will show that this subspace is equal to $\mathrm{Ran}(\Omega)^\perp$.

With these definitions, we may state our main theorem:

Theorem 3.2. *Let $H = H_0 + V$ where $H_0 = -\frac{1}{2}\Delta$ and V is a real-valued multiplication operator such that*

- $V \in L^\infty(\mathbb{R}^d)$
- *There exists a collection of cones $\{\mathcal{C}_i\}_{i\in\mathcal{I}}$ for which V satisfies the* generalized Enss condition

$$\|\chi_{\mathcal{A}_r} V\|_{\rm op} \in L^1([0,\infty), dr) \tag{3.1}$$

Then

(i) (Existence) For all $\psi \in \mathcal{D}$ the limit $\Omega\psi$ exists. Furthermore, $\sigma(H_0) \subset \sigma_{\rm ac}(H)$.

(ii) (Dynamical description of scattering states and their complement) We have

$$\Omega(\mathcal{D}) = \overline{\mathcal{H}_{\rm scat}}$$
$$\Omega(\mathcal{D})^\perp = \mathcal{H}_{\rm int}$$

Furthermore, we also show that for half-spaces there are spatial characterizations of $\Omega(\mathcal{D})$ and $\Omega(\mathcal{D})^\perp$:

Theorem 3.3. *Suppose that $\{\mathcal{C}_i\}_{i\in\mathcal{I}}$ consists of half-spaces. Then we have that*

$$\Omega(\mathcal{D}) = \overline{\{\psi \in \mathcal{H} \mid \exists v > 0, \lim_{t\to\infty} \|\chi_{\mathcal{A}_{vt}^c}\psi_t\| = 0\}}$$
$$\Omega(\mathcal{D})^\perp = \{\psi \in \mathcal{H} \mid \forall v > 0, \lim_{t\to\infty} \|\chi_{\mathcal{A}_{vt}}\psi_t\| = 0\}$$

Remark 3.4. In fact, using a small variation of the proofs below, one can give slightly different descriptions in Theorem 3.2:

$$\Omega(\mathcal{D}) = \{\psi \in \mathcal{H} \mid \forall n > 0, \exists m, \delta_0 > 0, \text{ so that } \forall \delta \in (0, \delta_0) \lim_{t\to\infty} \|(P_\delta(\mathcal{W}_{n,m;\rm out}) - \mathrm{Id})\psi_t\| = 0\}$$
$$\Omega(\mathcal{D})^\perp = \{\psi \in \mathcal{H} \mid \forall m, n > 0, \exists \delta_0 > 0 \text{ so that } \forall \delta \in (0, \delta_0) \lim_{t\to\infty} \|P_\delta(\mathcal{W}_{n,m;\rm out})\psi_t\| = 0\}$$

In this description, the space and time variables are decoupled completely, which gives some further insight into the behavior of the interacting states. Since Theorem 3.3 requires taking $n = vt$, we use the corresponding microlocal definition in Theorem 3.2.

4. Existence of the Wave Operator Ω

First, we record a geometric fact:

Proposition 4.1. *We may write $A_r(\mathcal{C}_{x,\vec{v},\gamma}) = \mathcal{C}_{x,\vec{v},\gamma} + \frac{r}{\sin(\gamma)}\vec{v}$.*

Proof. By projecting to any plane containing $\vec{v}$, the claim is clear from Figure 1. □

For $r \leq 0$ we will use the above as the definition of $A_r(\mathcal{C})$. We now use the following direct application of the Corollary to Theorem XI.14 from [13]:

Lemma 4.2. *Let $u \in \mathcal{S}$ and let $\mathcal{G}$ be an open set such that* $\operatorname{supp} \hat{u} \Subset \mathcal{G}$ *. Then for any $\ell \in \mathbb{N}$, there is a constant $C > 0$ depending on ℓ, u, and $\mathcal{G}$ so that*

$$|e^{-itH_0}u(x)| \leq C(1 + \|x\| + |t|)^{-\ell}$$

for all pairs (x, t) such that $\frac{x}{t} \notin \mathcal{G}$.

We let

$$\mathcal{D}_k(\mathcal{C}_{x,\vec{v},\gamma}) = \{\psi \in \mathcal{S} \mid \operatorname{supp} \hat{\psi} \Subset A_k(\mathcal{C}_{\vec{v},\gamma})\}$$

Note that the set $\mathcal{D}_k(\mathcal{C}_{x,\vec{v},\gamma})$ is independent of the vertex x and that $\bigcup_{k>0} \mathcal{D}_k(\mathcal{C}_{x,\vec{v},\gamma})$ is dense in $\mathcal{D}(\mathcal{C}_{x,\vec{v},\gamma})$.

We use this to prove the following proposition which will be useful here and in the sequel.

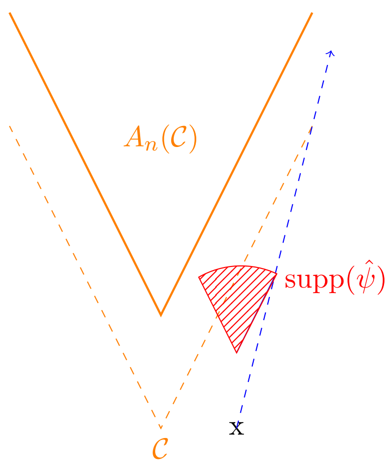

FIGURE 4. Illustration of the momentum of $\hat{\psi}$, in red, with respect to $A_n(\mathcal{C})$, in orange. The dashed blue line corresponds to a classic trajectory from x with momentum at the edge of the red cone.

Proposition 4.3. *Let $\mathcal{C}$ be any cone and $k > 0$. Then there exists $c(\mathcal{C}) > 0$ such that for all $\psi \in \mathcal{D}_k(\mathcal{C})$ and any $\ell > 0$ there exists $C(\psi, \ell) > 0$ such that*

$$\|\chi_{A_n^c(\mathcal{C})} e^{-itH_0}\psi\| \leq C(1 + |t|)^{-\ell}$$

for any pair of $(n, t) \in \mathbb{R}^2$ satisfying

$$c < kt - n \tag{4.1}$$

Proof. Write $\mathcal{C} = \mathcal{C}_{x,\vec{v},\gamma}$. In order to apply Lemma 4.2, we take $\mathcal{G} = A_k(\mathcal{C}_{\vec{v},\gamma})$. Thus, we must show that so long as $kt - n$ is sufficiently large, for all $y \in A_n^c(\mathcal{C}_{x,\vec{v},\gamma})$, we have that $\frac{y}{t} \in A_k^c(\mathcal{C}_{\vec{v},\gamma})$ or equivalently

$$tA_k(\mathcal{C}_{\vec{v},\gamma}) \subset A_n(\mathcal{C}_{x,\vec{v},\gamma})$$

Using Proposition 4.1 and the fact that $\mathcal{C}_{\vec{v},\gamma}$ is invariant under scaling, we see that we must show that

$$\mathcal{C}_{\vec{v},\gamma} + \frac{kt}{\sin(\gamma)}\vec{v} \subset x + \mathcal{C}_{\vec{v},\gamma} + \frac{n}{\sin(\gamma)}\vec{v} \iff \mathcal{C}_{\vec{v},\gamma} + \frac{kt-n}{\sin(\gamma)}\vec{v} - x \subset \mathcal{C}_{\vec{v},\gamma} \tag{4.2}$$

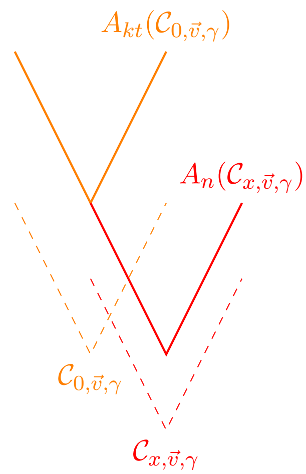

FIGURE 5. Illustration of the inclusion (4.2)

In words, we must show that the cone $\mathcal{C}_{\vec{v},\gamma}$ shifted by the vector $\frac{kt-n}{\sin(\gamma)}\vec{v} - x$ is contained in $\mathcal{C}_{\vec{v},\gamma}$, which will be the case as long as this vector lies in the cone. But this is clearly true if $kt - n$ is large enough with respect to fixed x i.e if (4.1) holds.

Therefore, we may apply Lemma 4.2, to see that for any $\ell > 0$

$$|e^{-itH_0}\psi(y)| \leq C(1 + \|y\| + |t|)^{-\ell}$$

for all $y \in A_n^c(\mathcal{C}_{x,\vec{v},\gamma})$ where C is independent of y and t. Choosing ℓ large enough, we get that

$$\|\chi_{A_n^c(\mathcal{C}_{x,\vec{v},\gamma})} e^{-itH_0}\psi\|^2 \leq C \int\limits_{A_n^c(\mathcal{C}_{x,\vec{v},\gamma})} (1 + \|y\| + |t|)^{-\ell}\, dx < C(1 + |t|)^{-\ell+d} \tag{4.3}$$

as needed. □

This is already enough to prove the existence of the wave operators:

Proof of part (i) of Theorem 3.2. By Cook's method (see [13] Theorem XI.4), it suffices to show that for all ψ in some dense subset of $\mathcal{D} = \overline{\bigcup\limits_{i\in\mathcal{I}} \mathcal{D}(\mathcal{C}_i)}$

$$\int\limits_0^\infty \|Ve^{-itH_0}\psi\|\, dt < \infty$$

We will take as our dense subset $\bigcup\limits_{i\in\mathcal{I}} \bigcup\limits_{k>0} \mathcal{D}_k(\mathcal{C}_i)$.

For any $i \in \mathcal{I}$ and any $k > 0$, write $\mathcal{C}_i = \mathcal{C}_{x,\vec{v},\gamma}$, let $0 < \varepsilon < k$, and let $\psi \in \mathcal{D}_k(\mathcal{C}_{x,\vec{v},\gamma})$. We can then write

$$\begin{aligned}\|Ve^{-itH_0}\psi\| &\leq \|V\chi_{\mathcal{A}_{\varepsilon t}} e^{-itH_0}\psi\| + \|V\chi_{\mathcal{A}_{\varepsilon t}^c} e^{-itH_0}\psi\| \\ &\leq \|V\chi_{\mathcal{A}_{\varepsilon t}}\|_{\text{op}}\|\psi\| + M\|\chi_{A_{\varepsilon t}^c(\mathcal{C}_i)} e^{-itH_0}\psi\|\end{aligned}$$

as $\mathcal{A}_{\varepsilon t}^c \subset A_{\varepsilon t}^c(\mathcal{C}_i)$. The first term is $L^1([0,\infty), dt)$ by the assumption (3.1) whereas we will estimate the second term via Proposition 4.3. For this, let $c_0 = c_0(\mathcal{C}_{x,\vec{v},\gamma})$ be the constant from Proposition 4.3 and let $T_0 = \frac{c_0}{k-\varepsilon}$. By Proposition 4.3 with $n = \varepsilon t$, we see that for any $\ell > 0$ and $t > T_0$ there is some $C > 0$ so that

$$\|\chi_{A_{\varepsilon t}^c(\mathcal{C}_i)} e^{-itH_0}\psi\| \leq C(1 + t)^{-\ell} \tag{4.4}$$

uniformly in t. It follows immediately that $\|Ve^{-itH_0}\psi\|$ is integrable on $[0,\infty)$ as needed.

Furthermore, since $\sigma(H_0|_{\mathcal{D}}) = \sigma(H_0)$, the intertwining property of Ω implies that $\sigma(H_0) \subset \sigma_{\mathrm{ac}}(H)$ as claimed. $\square$

5. Descriptions of $\mathrm{Ran}(\Omega)$ and $\mathrm{Ran}(\Omega)^{\perp}$

In this section, we give descriptions of $\mathrm{Ran}(\Omega)$ and its orthogonal complement in terms of the dynamics of H. In particular, we show that states in these subspaces may be characterized by their location in phase space as $t \to \infty$. Here, as before, $\mathrm{Ran}(\Omega)$ indicates the range of Ω on its natural domain $\mathcal{D}$, as defined in the previous section.

5.1. **Characterizing** $\mathrm{Ran}(\Omega)$**.** Recall the definition of the outgoing set associated with a single cone $\mathcal{C}_{x,\vec{v},\gamma}$

$$W_{n;\mathrm{out}}(\mathcal{C}_{x,\vec{v},\gamma}) = \{(y,p) \in \mathbb{R}^{2d} \mid y \in A_n(\mathcal{C}_{x,\vec{v},\gamma}) \text{ and } p \in \mathcal{C}_{\vec{v},\gamma}\}$$

and that $\mathcal{W}_{n,m;\mathrm{out}} = \bigcup_{i\in\mathcal{I}} W_{n,m;\mathrm{out}}(\mathcal{C}_i)$.

We again record some purely geometric facts:

Proposition 5.1. *Let $\mathcal{C}$ be any convex cone and let $\mathcal{A}_r^c$ be defined relative to a collection of cones that contains $\mathcal{C}$.*

(1) For any $n,t,r \geq 0$ and $m \in \mathbb{R}$

$$d(\mathfrak{C}_t(W_{n,m;\mathrm{out}}(\mathcal{C})), A_r^c(\mathcal{C})) \geq n + mt - r$$

(2) For any $n,t,r \geq 0$ and $m \in \mathbb{R}$

$$d(\mathfrak{C}_t(W_{n,m;\mathrm{out}}(\mathcal{C})), \mathcal{A}_r^c) \geq n + mt - r$$

Proof. The proof of (1) follows from the fact that for any $t \geq 0$

$$A_n(\mathcal{C}_{x,\vec{v},\gamma}) + tA_m(\mathcal{C}_{\vec{v},\gamma}) = A_{n+tm}(\mathcal{C}_{x,\vec{v},\gamma})$$

Indeed, because $t\mathcal{C}_{\vec{v},\gamma} = \mathcal{C}_{\vec{v},\gamma}$ and $A_n(\mathcal{C}_{x,\vec{v},\gamma}) = x + \mathcal{C}_{\vec{v},\gamma} + \frac{n}{\sin(\gamma)}\vec{v}$ we see that

$$\begin{aligned} A_n(\mathcal{C}_{x,\vec{v},\gamma}) + tA_m(\mathcal{C}_{\vec{v},\gamma}) &= (x + \mathcal{C}_{\vec{v},\gamma} + \frac{n}{\sin(\gamma)}\vec{v}) + t(\mathcal{C}_{\vec{v},\gamma} + \frac{m}{\sin(\gamma)}\vec{v}) \\ &= x + \mathcal{C}_{\vec{v},\gamma} + (\frac{n}{\sin(\gamma)} + \frac{tm}{\sin(\gamma)})\vec{v} = A_{n+tm}(\mathcal{C}_{x,\vec{v},\gamma}) \end{aligned} \tag{5.1}$$

where we have used that $\mathcal{C}_{\vec{v},\gamma}$ is convex. The claim now follows from the definition of $A_r(\mathcal{C})$.

The proof of (2) is now immediate because for all $i \in \mathcal{I}$, $\mathcal{A}_r^c \subset A_r^c(\mathcal{C}_i)$. $\square$

With this in hand, we can prove the main technical estimate in the proof of Theorem 3.2 (ii).

Lemma 5.2. *For any $v > 0$, $\delta < \frac{v}{2}$, $0 < \varepsilon < \frac{v}{2} - \delta$, and $\ell > 0$ there exists $C > 0$ so that for any $2t \geq s \geq t > 0$*

$$\|(\Omega(s-t) - \mathrm{Id})P_\delta(\mathcal{W}_{vt;\mathrm{out}})\|_{\mathrm{op}} \leq \int_t^s \|V\chi_{\mathcal{A}_{\varepsilon w}}\|_{\mathrm{op}}\, dw + Ct^{-\ell} \tag{5.2}$$

For any v, m and $\ell > 0$ if $0 < \delta < m$, and $0 < \varepsilon < m - \delta$ there exists $C > 0$ so that for any $s \geq t > 0$

$$\|(\Omega(s-t) - \mathrm{Id})P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\|_{\mathrm{op}} \leq \int_t^\infty \|V\chi_{\mathcal{A}_{\varepsilon w}}\|_{\mathrm{op}}\, dw + Ct^{-\ell} \tag{5.3}$$

Proof. We start by noting that

$$\|(\Omega(s-t)-\mathrm{Id})P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\|_{\mathrm{op}} = \|(\Omega(s)-\Omega(t))P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\|_{\mathrm{op}}$$

We use the identity

$$\Omega(s)-\Omega(t) = \int_t^s e^{iwH} i(H-H_0)e^{-iwH_0}\,dw$$

by writing, for $0<\varepsilon<\frac{v}{2}-\delta$

$$\|\int_t^s e^{iwH}(H-H_0)e^{-iwH_0}P_\delta(\mathcal{W}_{vt;\mathrm{out}})\,dw\|_{\mathrm{op}} \le \int_t^s \|Ve^{-iwH_0}P_\delta(\mathcal{W}_{vt;\mathrm{out}})\|_{\mathrm{op}}\,dw$$

$$\le \int_t^s \|V\chi_{\mathcal{A}_{\varepsilon w}}\|_{\mathrm{op}}\,dw + M\int_t^s \|\chi_{\mathcal{A}^c_{\varepsilon w}}e^{-iwH_0}P_\delta(\mathcal{W}_{vt;\mathrm{out}})\|_{\mathrm{op}}\,dw$$

From the microlocal non-stationary phase estimate on P_δ (Lemma A.5) and Proposition 5.1 with $m=0$, we see that

$$\|\chi_{\mathcal{A}^c_{\varepsilon w}}e^{-iwH_0}P_\delta(\mathcal{W}_{vt;\mathrm{out}})\|_{\mathrm{op}} < C[vt-w\varepsilon]^{-(\ell+1)} < C[(v-2\varepsilon)t]^{-(\ell+1)}$$

for all $w<2t$ since by Proposition 5.1 we have

$$d(\mathfrak{C}_w(\mathcal{W}_{vt;\mathrm{out}}),\mathcal{A}^c_{\varepsilon w}) \ge vt-w\varepsilon > (\frac{v}{2}-\varepsilon)w$$

which is in turn greater than δw because $\varepsilon<\frac{v}{2}-\delta$. Therefore, because $s-t\le t$

$$\int_t^s \|\chi_{\mathcal{A}^c_{\varepsilon w}}e^{-iwH_0}P_\delta(\mathcal{W}_{vt;\mathrm{out}})\|_{\mathrm{op}}\,dw \le C\frac{s-t}{(v-2\varepsilon)^{\ell+1}t^{\ell+1}} \le Ct^{-\ell} \tag{5.4}$$

In summary, we see that for any $t>0$ and any $s\in[t,2t]$

$$\|(\Omega(s-t)-\mathrm{Id})P_\delta(\mathcal{W}_{vt;\mathrm{out}})\|_{\mathrm{op}} \le \int_t^s \|V\chi_{\mathcal{A}_{\varepsilon w}}\|_{\mathrm{op}}\,dw + Ct^{-\ell}$$

for some constant C independent of t and s.

If we replace $\mathcal{W}_{vt;\mathrm{out}}$ with $\mathcal{W}_{vt,m;\mathrm{out}}$, again from Lemma A.5 and Proposition 5.1 with $m>0$, we see that, for $\varepsilon<m-\delta$

$$\|\chi_{\mathcal{A}^c_{\varepsilon w}}e^{iwH_0}P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\|_{\mathrm{op}} < C(vt+(m-\varepsilon)w)^{-(\ell+1)}$$

for all $w>0$ since by Proposition 5.1, for $\delta<m$ and $\varepsilon<m-\delta$

$$d(\mathfrak{C}_w(\mathcal{W}_{vt,m;\mathrm{out}}),\mathcal{A}^c_{\varepsilon w}) = vt+mw-\varepsilon w > (m-\varepsilon)w > \delta w$$

Therefore,

$$\int_t^s \|\chi_{\mathcal{A}^c_{\varepsilon w}}e^{iwH_0}P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\|_{\mathrm{op}}\,dw \le C\int_t^s (vt+(m-\varepsilon)w)^{-\ell-1}\,dw \le C(vt)^{-\ell}$$

In summary, we see that for any $s>t>0$

$$\|(\Omega(s-t)-\mathrm{Id})P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\|_{\mathrm{op}} \le \int_t^\infty \|V\chi_{\mathcal{A}_{\varepsilon w}}\|_{\mathrm{op}}\,dw + Ct^{-\ell}$$

for some constant C independent of t and s.

□

Recall that

$$\mathcal{H}_{\mathrm{scat}} = \{\psi \in \mathcal{H} \mid \exists v, m, \delta_0 > 0 \text{ so that } \forall \delta \in (0, \delta_0) \lim_{t\to\infty} \|(P_\delta(\mathcal{W}_{vt,m;\mathrm{out}}) - \mathrm{Id})\psi_t\| = 0\}$$

Theorem 5.3. *Let $\psi \in \mathcal{H}_{\mathrm{scat}}$. Then $\Omega^*\psi$ exists and is in $\mathcal{D}$, or equivalently $\psi \in \Omega(\mathcal{D})$.*

Alternatively, if for some $v > 0, \delta < \frac{v}{2}$ and $\ell > 1$ there exists $C > 0$ so that

$$\|(P_\delta(\mathcal{W}_{vt;\mathrm{out}}) - \mathrm{Id})\psi_t\| < Ct^{-\ell}$$

for all $t > 0$ then $\Omega^*\psi$ exists and lies in $\mathcal{D}$.

Proof. We show that

$$\Omega^*(t) := e^{itH_0}e^{-itH}$$

is Cauchy as $t \to \infty$. For that, fix $t > 0$ and suppose that $t \le s$. Observe that

$$\|(\Omega^*(s) - \Omega^*(t))\psi\| = \|(\Omega(s-t) - \mathrm{Id})\psi_t\|$$

which comes from multiplying by $e^{-itH}\Omega(s)$ and the identity

$$e^{-itH}\Omega(s)\Omega^*(t) = \Omega(s-t)e^{-itH}$$

Since $\psi \in \mathcal{H}_{\mathrm{scat}}$, there is some $v, m, \delta_0 > 0$ such that for any $\delta < \delta_0$

$$\|(P_\delta(\mathcal{W}_{vt,m;\mathrm{out}}) - \mathrm{Id})\psi_t\| = o(1)$$

as $t \to \infty$. For these $v, m > 0$ choose $\delta < \min(m, \delta_0)$, so we may write

$$\begin{aligned}\|(\Omega(s-t) - \mathrm{Id})\psi_t\| &\le \|(\Omega(s-t) - \mathrm{Id})P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t\| + \|(\Omega(s-t) - \mathrm{Id})(P_\delta(\mathcal{W}_{vt,m;\mathrm{out}}) - \mathrm{Id})\psi_t\| \\ &= \|(\Omega(s-t) - \mathrm{Id})P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t\| + o(1)\end{aligned}$$

as $t \to \infty$. By using Lemma 5.2, we conclude that, for $0 < \varepsilon < m - \delta$

$$\|(\Omega^*(s) - \Omega^*(t))\psi\| \le Ct^{-\ell} + \int_t^\infty \|V\chi_{\mathcal{A}_{\varepsilon w}}\|_{\mathrm{op}}\, dw \|\psi\| + o(1)$$

for some constant C independent of t and s. The second term decays with t, by assumption (3.1), and thus the entire expression goes to 0 as $t \to \infty$.

This shows that $\Omega^*\psi$ exists, so to see that it lies in $\mathcal{D}$ first note that for $\delta < m$, $P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t \in \mathcal{D}$ by Proposition A.3 and thus so does $e^{itH_0}P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t$. Now observe

$$\|\Omega^*(t)\psi - e^{itH_0}P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t\| = \|\psi_t - P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t\| \xrightarrow{t\to\infty} 0$$

so that the claim follows because $\mathcal{D}$ is closed.

To see the second claim, for $t > 0$ and $s \in [t, 2t]$, write as before, for the given v and $\delta < \frac{v}{2}$

$$\begin{aligned}\|(\Omega(s-t) - \mathrm{Id})\psi_t\| &\le \|(\Omega(s-t) - \mathrm{Id})P_\delta(\mathcal{W}_{vt;\mathrm{out}})\psi_t\| + \|(\Omega(s-t) - \mathrm{Id})(P_\delta(\mathcal{W}_{vt;\mathrm{out}}) - \mathrm{Id})\psi_t\| \\ &\le \|(\Omega(s-t) - \mathrm{Id})P_\delta(\mathcal{W}_{vt;\mathrm{out}})\psi_t\| + Ct^{-\ell}\end{aligned}$$

and again apply Lemma 5.2 to see that

$$\|(\Omega^*(s) - \Omega^*(t))\psi\| \le Ct^{-\ell} + \int_t^s \|V\chi_{\mathcal{A}_{\varepsilon w}}\|_{\mathrm{op}}\, dw \|\psi\|$$

for some constant C independent of t and s.

To conclude, for any $s \geq t > 0$, fix N so that $s \in [2^N t, 2^{N+1}t]$ and then write

$$\begin{aligned}\|(\Omega^*(s) - \Omega^*(t))\psi\| &\leq \sum_{n=0}^{N-1} \|(\Omega^*(2^{n+1}t) - \Omega^*(2^n t))\psi\| + \|(\Omega^*(s) - \Omega^*(2^N t))\psi\| \\ &\leq C\sum_{n=0}^{N}(2^n t)^{-\ell} + \sum_{n=0}^{N-1}\int_{2^n t}^{2^{n+1}t} \|V\chi_{\mathcal{A}_{\varepsilon w}}\|_{\mathrm{op}}\, dw\|\psi\| + \int_{2^N t}^{s} \|V\chi_{\mathcal{A}_{\varepsilon w}}\|_{\mathrm{op}}\, dw\|\psi\| \\ &\leq Ct^{-\ell} + \int_t^s \|V\chi_{\mathcal{A}_{\varepsilon w}}\|_{\mathrm{op}}\, dw\|\psi\|\end{aligned}$$

where C does not depend on N. Note that in this step we require the prescribed rate of convergence in $\|(P_\delta(\mathcal{W}_{vt,m;\mathrm{out}}) - \mathrm{Id})\psi_t\| < Ct^{-\ell}$. This, combined with condition (3.1), prove that $\Omega^*\psi$ exists.

To see that $\Omega^*\psi$ lies in $\mathcal{D}$, we proceed as before by noting that for any $\delta > 0$, Proposition A.3 shows that

$$\operatorname{supp} \mathcal{F}(P_\delta(\mathcal{W}_{vt;\mathrm{out}})\psi) \subset B_\delta + \bigcup_{(x,\vec{v},\gamma)\in\mathcal{I}} \mathcal{C}_{\vec{v},\gamma}$$

Now we can write

$$\|\Omega^*\psi - e^{itH_0}P_\delta(\mathcal{W}_{vt;\mathrm{out}})\psi_t\| \leq \|\Omega^*\psi - \Omega^*(t)\psi\| + \|\Omega^*(t)\psi - e^{itH_0}P_\delta(\mathcal{W}_{vt;\mathrm{out}})\psi_t\|$$

By taking the limit $t \to \infty$ we see that for any $v > 2\delta > 0$

$$\operatorname{supp} \widehat{\Omega^*\psi} \subset \overline{B_\delta + \bigcup_{(x,\vec{v},\gamma)\in\mathcal{I}} \mathcal{C}_{\vec{v},\gamma}}$$

Varying over all $\delta > 0$, we conclude that $\Omega^*\psi \in \mathcal{D}$ because $\mathcal{D} = \overline{\bigcup_{i\in\mathcal{I}} \mathcal{D}(\mathcal{C}_i)}$. ☐

Having shown that $\mathcal{H}_{\mathrm{scat}} \subset \mathrm{Ran}(\Omega)$, we now show that $\mathcal{H}_{\mathrm{scat}}$ is dense in this subspace. For this we will start with a lemma:

Lemma 5.4. *Let $\psi \in \Omega(\mathcal{D}_k(\mathcal{C}_i))$ for some $i \in \mathcal{I}$ and $k > 0$. Then there is some $T_0 = T_0(k, \mathcal{C}_i)$ such that for any v, m, ε, ℓ, and δ satisfying*

$$v, \varepsilon \in (0, k) \qquad 0 \leq m < k \qquad 0 < \delta < \min(k - m, \frac{k - v}{2}) \qquad \ell > 0$$

there exists $C > 0$ such that

$$\|(P_\delta(\mathcal{W}_{vt,m;\mathrm{out}}) - \mathrm{Id})\psi_t\| \leq Ct^{-\ell} + \int_t^\infty \|\chi_{A_{\varepsilon s}}V\|_{\mathrm{op}}\|\psi\|\, ds \tag{5.5}$$

for all $t > T_0$.

Proof. Let $\psi = \Omega\varphi$ for $\varphi \in \mathcal{D}_k(\mathcal{C}_i)$, some $i \in \mathcal{I}$, and some fixed $k > 0$. It suffices to show that for some choice of parameters as above that for all $t > T_0$

$$\|(P_\delta(\mathcal{W}_{vt,m;\mathrm{out}}) - \mathrm{Id})e^{-itH_0}\varphi\| < Ct^{-\ell} \tag{5.6}$$

and

$$\|(\Omega - \mathrm{Id})e^{-itH_0}\varphi\| \leq Ct^{-\ell} + \int_t^\infty \|\chi_{A_{\varepsilon s}}V\|_{\mathrm{op}}\|\psi\|\, ds \tag{5.7}$$

in light of the inequality

$$\begin{aligned}\|(P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})-\mathrm{Id})e^{-itH}\psi\| &= \|(P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})-\mathrm{Id})\Omega e^{-itH_0}\varphi\| \\ &\leq \|(P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})-\mathrm{Id})e^{-itH_0}\varphi\| + \|(P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})-\mathrm{Id})(\Omega-\mathrm{Id})e^{-itH_0}\varphi\|\end{aligned}$$

and the fact that $\|P_\delta(W_{vt,m;\mathrm{out}})-\mathrm{Id}\,\|_{\mathrm{op}}$ is bounded independently of t.

The inequality (5.7) is proven by first choosing $\varepsilon < k$ and writing

$$\begin{aligned}\|(\Omega-\mathrm{Id})e^{-itH_0}\varphi\| &\leq \int_0^\infty \|Ve^{-i(s+t)H_0}\varphi\|\,ds = \int_t^\infty \|Ve^{-isH_0}\varphi\|\,ds \\ &\leq \int_t^\infty \|\chi_{\mathcal{A}_{\varepsilon s}}V\|_{\mathrm{op}}\|\psi\|\,ds + M\int_t^\infty \|\chi_{\mathcal{A}^c_{\varepsilon s}}e^{-isH_0}\varphi\|\,ds\end{aligned}$$

where have used that $\|\varphi\| = \|\psi\|$. Now let c be the constant from Proposition 4.3 and note that

$$c < ks - \varepsilon s$$

so long as $s > T_1 := \frac{c}{k-\varepsilon}$. Therefore, Proposition 4.3 with $n = \varepsilon s$ implies the desired inequality for any $t > T_1$. Therefore, it remains to show (5.6) for some choice of parameters as above.

To see inequality (5.6), we write $\mathcal{C}_i = \mathcal{C}_{x,\vec{v},\gamma}$ and observe that

$$\begin{aligned}\|(P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})-\mathrm{Id})e^{-itH_0}\varphi\|^2 &\leq \langle e^{-itH_0}\varphi, P_\delta(\mathcal{W}^c_{vt,m;\mathrm{out}})e^{-itH_0}\varphi\rangle \\ &\leq \langle e^{-itH_0}\varphi, P_\delta(W^c_{vt,m;\mathrm{out}}(\mathcal{C}_i))e^{-itH_0}\varphi\rangle \leq \|\varphi\|\|P_\delta(W^c_{vt,m;\mathrm{out}}(\mathcal{C}_i))e^{-itH_0}\varphi\|\end{aligned}$$

Noting that

$$W^c_{vt,m;\mathrm{out}}(\mathcal{C}_i) = A^c_{vt}(\mathcal{C}_i)\times\mathbb{R}^d \bigsqcup A_{vt}(\mathcal{C}_i)\times A^c_m(\mathcal{C}_{\vec{v},\gamma})$$

and recalling that $\operatorname{supp}\hat{\varphi} \Subset A_k(\mathcal{C}_{\vec{v},\gamma})$, by the momentum localization properties of P_δ (Proposition A.3) we see that

$$P_\delta(W^c_{vt,m;\mathrm{out}}(\mathcal{C}_i))\varphi = P_\delta(A^c_{vt}(\mathcal{C}_i)\times\mathbb{R}^d)\varphi$$

as $\delta < k - m$ and $d(A_k(\mathcal{C}_{\vec{v},\gamma}), A^c_m(\mathcal{C}_i)) > k - m$.

Next, choose $T_2 = \frac{2\|x\|}{k-v}$ so that for any $t > T_2$

$$d(A^c_{vt}(\mathcal{C}_i), tA_k(\mathcal{C}_{\vec{v},\gamma})) > (k-v)t - \|x\| > \frac{k-v}{2}t > \delta t$$

Proposition A.6 then implies that for any $\ell > 0$ there is some $C > 0$ such that

$$\|P_\delta(W^c_{vt,m;\mathrm{out}}(\mathcal{C}_i))e^{-itH_0}\varphi\| < Ct^{-\ell}$$

for all $t > T_2$ which proves (5.7). We then conclude that the lemma holds with $T_0 = \max(T_1, T_2)$. $\square$

Theorem 5.5. *Suppose that* $\psi \in \operatorname{Ran}(\Omega)$. *Then* $\psi \in \overline{\mathcal{H}_{\mathrm{scat}}}$.

Proof. Since $\bigcup_{i\in\mathcal{I}}\bigcup_{k>0}\Omega(\mathcal{D}_k(\mathcal{C}_i))$ is dense in $\operatorname{Ran}(\Omega)$, it suffices to show that $\Omega(\mathcal{D}_k(\mathcal{C}_i)) \subset \mathcal{H}_{\mathrm{scat}}$ for all $k > 0$. But this is immediate from Lemma 5.4 so long as m, ε and v are chosen appropriately with respect to k and δ_0 is chosen to be less than $\min(k-m, \frac{k-v}{2})$. $\square$

Remark 5.6. The second claim in Theorem 5.3 and the above proof of Theorem 5.5 also show that $\operatorname{Ran}(\Omega)$ may be described without the parameter m as

$$\Omega(\mathcal{D}) = \overline{\{\psi\in\mathcal{H} \mid \exists v, C, \ell, \delta_0 > 0 \text{ so that } \forall\delta\in(0,\delta_0) \text{ and } t > 0\, \|(P_\delta(\mathcal{W}_{vt;\mathrm{out}})-\mathrm{Id})\psi_t\| < Ct^{-\ell}\}}$$

but we prefer the given characterization as $\mathcal{H}_{\mathrm{scat}}$ because $\mathcal{H}_{\mathrm{int}}$ must be defined in terms of m.

5.2. **Characterizing** $\mathrm{Ran}(\Omega)^\perp$**.** Recall that

$$\mathcal{H}_{\mathrm{int}} = \{\psi \in \mathcal{H} \mid \forall v, m > 0, \exists \delta_0 > 0 \text{ so that } \forall \delta \in (0, \delta_0) \lim_{t\to\infty} \|P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t\| = 0\}$$

Theorem 5.7. *Under the above definition,* $\mathrm{Ran}(\Omega)^\perp = \mathcal{H}_{\mathrm{int}}$.

Proof. For the inclusion $\mathrm{Ran}(\Omega)^\perp \subset \mathcal{H}_{\mathrm{int}}$, take $\psi \in \mathrm{Ran}(\Omega)^\perp$ and fix any $v, m > 0$ and $\delta < m$. For any $s > t > 0$ we may write

$$\begin{aligned}
&\|P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t\|^2 \le \langle P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t, \psi_t\rangle \\
&\le \langle \Omega(s-t)P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t, \psi_t\rangle + \|\psi\|\|(\Omega(s-t) - \mathrm{Id})P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t\| \\
&= \langle e^{itH}\Omega(s-t)P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t, \psi\rangle + \|\psi\|\|(\Omega(s-t) - \mathrm{Id})P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t\| \\
&= \langle \Omega(s)e^{itH_0}P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t, \psi\rangle + \|\psi\|\|(\Omega(s-t) - \mathrm{Id})P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t\|
\end{aligned}$$

where we have used that

$$e^{itH}\Omega(s-t) = \Omega(s)e^{itH_0}$$

Now by applying (5.3) from Lemma 5.2 to the second term, we get that for any $s \ge t > 0$, and $0 < \varepsilon < m - \delta$

$$\|P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t\|^2 \le \langle \Omega(s)e^{itH_0}P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t, \psi\rangle + Ct^{-\ell} + \int_t^\infty \|V\chi_{\mathcal{A}_{\varepsilon w}}\|_{\mathrm{op}}\, dw$$

for some constant C that does not depend on t or s. Observe that because $\delta < m$, by Proposition A.3, $P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t$ lies in $\mathcal{D}$ as does $e^{itH_0}P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t$ since the free propagator does not alter the momentum support of a state. Thus, with t fixed, we may take the limit $s \to \infty$ in the above to obtain

$$\begin{aligned}
\|P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t\|^2 &\le \langle \Omega e^{itH_0}P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t, \psi\rangle + Ct^{-\ell} + \int_t^\infty \|V\chi_{\mathcal{A}_{\varepsilon w}}\|_{\mathrm{op}}\, dw \\
&= Ct^{-\ell} + \int_t^\infty \|V\chi_{\mathcal{A}_{\varepsilon w}}\|_{\mathrm{op}}\, dw \xrightarrow{t\to\infty} 0
\end{aligned}$$

by assumption (3.1) and the fact that $\psi \perp \mathrm{Ran}(\Omega)$. This proves the first inclusion.

Conversely, let $\psi \in \mathcal{H}_{\mathrm{int}}$. We will show that $\psi \perp \Omega(\mathcal{D}_k(\mathcal{C}_i))$ for any $k > 0, i \in \mathcal{I}$ and conclude by density. Let $\varphi \in \Omega(\mathcal{D}_k(\mathcal{C}_i))$ for some $k > 0, i \in \mathcal{I}$ and let m, ε and v satisfy $m, v, \varepsilon \in (0, k)$. Then by Lemma 5.4 for δ sufficiently small there exists some $T_0(\mathcal{C}_i) > 0$ such that there are constants $C > 0$ and $\ell > 0$ so that

$$\|(P_\delta(\mathcal{W}_{vt,m;\mathrm{out}}) - \mathrm{Id})\varphi_t\| < Ct^{-\ell} + \int_t^\infty \|\chi_{A_{\varepsilon s}}V\|_{\mathrm{op}}\|\varphi\|\, ds$$

for all $t > T_0$.

Then we have that for any $t > T_0$

$$\begin{aligned}
\langle \psi, \varphi\rangle &= \langle P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t, \varphi\rangle + \langle \psi_t, (P_\delta(\mathcal{W}_{vt,m;\mathrm{out}}) - \mathrm{id})\varphi_t\rangle \\
&\le \|P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t\|\|\varphi\| + \|\psi\|\|(P_\delta(\mathcal{W}_{vt,m;\mathrm{out}}) - \mathrm{Id})\varphi_t\| \\
&< \|P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t\|\|\varphi\| + Ct^{-\ell} + \int_t^\infty \|\chi_{A_{\varepsilon s}}V\|_{\mathrm{op}}\|\varphi\|\, ds
\end{aligned}$$

Since $\psi \in \mathcal{H}_{\text{int}}$, for the same v and m, and choosing δ smaller if necessary, we have that

$$\lim_{t\to\infty} \|P_\delta(\mathcal{W}_{vt,m;\text{out}})\psi_t\| = 0$$

so we may conclude that

$$\langle \psi, \varphi \rangle < \|P_\delta(\mathcal{W}_{vt,m;\text{out}})\psi_t\|\|\varphi\| + Ct^{-\ell} + \int_t^\infty \|\chi_{A_{\varepsilon s}} V\|_{\text{op}}\|\varphi\|\, ds \xrightarrow{t\to\infty} 0$$

from assumption (3.1). Therefore, $\psi \perp \varphi$, as needed. □

5.3. **Spatial characterizations of $\mathcal{H}_{\text{scat}}$ and $\mathcal{H}_{\text{int}}$.** In this section, we show that for some systems one can replace the microlocal descriptions of $\mathcal{H}_{\text{scat}}$ and $\mathcal{H}_{\text{int}}$ with descriptions that are purely spatial. Recall Theorem 3.3:

Theorem 3.3. *Suppose that $\{\mathcal{C}_i\}_{i\in\mathcal{I}}$ consists of half-spaces. Then with $\mathcal{D} = \overline{\bigcup_{i\in\mathcal{I}} \mathcal{D}(C_i)}$ we have that*

$$\Omega(\mathcal{D}) = \overline{\{\psi \in \mathcal{H} \mid \exists v > 0, \lim_{t\to\infty} \|\chi_{\mathcal{A}_{vt}^c}\psi_t\| = 0\}}$$
$$\Omega(\mathcal{D})^\perp = \{\psi \in \mathcal{H} \mid \forall v > 0, \lim_{t\to\infty} \|\chi_{\mathcal{A}_{vt}}\psi_t\| = 0\}$$

Remark 5.8. The above theorem applies to potentials for which $\{\mathcal{C}_i\}_{i\in\mathcal{I}}$ also contains cones of aperture less than π. In this case, one will have a spatial characterization only for those cones of large enough aperture. See Example 2.5 for one such setting.

So far, we have described the set of scattering states $\mathcal{H}_{\text{scat}}$ as those states which asymptotically propagate into some cone $\mathcal{C}$ with outgoing momenta, that is, those that point into $\mathcal{C}$. To obtain a spatial characterization, it suffices to show that it is impossible for a state to propagate into $\mathcal{C}$ with any other momentum localization if $\gamma = \frac{\pi}{2}$. For this, we begin by defining the incoming subset of phase space for any collection of cones: let

$$W_{n,m;\text{in}}(\mathcal{C}_{x,\vec{v},\gamma}) = \{(y,p) \in \mathbb{R}^{2d} \mid y \in A_n(\mathcal{C}_{x,\vec{v},\gamma}), -p \in A_{-m}(\mathcal{C}_{\vec{v},\gamma})\}$$
$$\mathcal{W}_{n,m;\text{in}} = \bigcup_{i\in\mathcal{I}} W_{n,m;\text{in}}(\mathcal{C}_i)$$

See Figure 6. We show that asymptotically no state can concentrate in these subsets of phase space:

Proposition 5.9. *For any $v > 0$, $0 < m < v$, and $\delta < \frac{v-m}{2}$*

$$\operatorname*{s-lim}_{t\to\infty} P_\delta(\mathcal{W}_{vt,m;\text{in}})e^{-itH} = 0 \tag{5.8}$$

Proof. This proof is based on an argument of Enss recorded in [16]. For any $\psi \in \mathcal{H}$ we can write

$$\|P_\delta(\mathcal{W}_{vt,m;\text{in}})e^{-itH}\psi\| \le \|P_\delta(\mathcal{W}_{vt,m;\text{in}})(e^{-itH} - e^{-itH_0})\psi\| + \|P_\delta(\mathcal{W}_{vt,m;\text{in}})e^{-itH_0}\psi\|$$

so to prove (5.8), it suffices to prove that for v, m, and δ as above

$$\lim_{t\to\infty} \|P_\delta(\mathcal{W}_{vt,m;\text{in}})(e^{-itH} - e^{-itH_0})\|_{\text{op}} = 0 \tag{5.9}$$

and

$$\operatorname*{s-lim}_{t\to\infty} P_\delta(\mathcal{W}_{vt,m;\text{in}})e^{-itH_0} = 0 \tag{5.10}$$

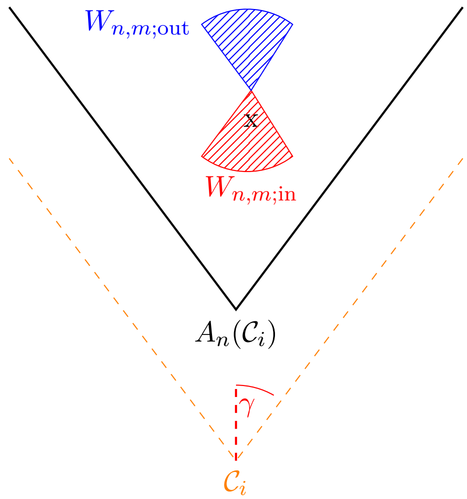

FIGURE 6. Illustration of the phase space sets $W_{n,m;\text{out}}(\mathcal{C}_i)$ and $W_{n,m;\text{in}}(\mathcal{C}_i)$: each has space coordinates inside the black cone with momentum coordinates inside the red/blue cone, respectively.

To prove (5.9), we write, for $\varepsilon < \frac{v-m}{4}$

$$\|P_\delta(\mathcal{W}_{vt,m;\text{in}})(e^{-itH} - e^{-itH_0})\|_{\text{op}} = \|(e^{itH} - e^{itH_0})P_\delta(\mathcal{W}_{vt,m;\text{in}})\|_{\text{op}}$$
$$= \|(\operatorname{Id} - e^{-itH}e^{itH_0})P_\delta(\mathcal{W}_{vt,m;\text{in}})\|_{\text{op}} \leq \int_0^t \|e^{-iwH}(-H+H_0)e^{iwH_0}P_\delta(\mathcal{W}_{vt,m;\text{in}})\|_{\text{op}}\,dw$$
$$\leq \int_0^t \|\chi_{\mathcal{A}_{\varepsilon(t+w)}}V\|_{\text{op}}\,dw + M\int_0^t \|\chi_{\mathcal{A}^c_{\varepsilon(t+w)}}e^{iwH_0}P_\delta(\mathcal{W}_{vt,m;\text{in}})\|_{\text{op}}\,dw$$
$$\leq \int_t^\infty \|\chi_{\mathcal{A}_{\varepsilon w}}V\|_{\text{op}}\,dw + M\int_0^t \|\chi_{\mathcal{A}^c_{\varepsilon(t+w)}}e^{iwH_0}P_\delta(\mathcal{W}_{vt,m;\text{in}})\|_{\text{op}}\,dw$$

Now we note that for any cone $\mathcal{C}_{x,\vec{v},\gamma}$

$$\begin{aligned}\mathfrak{C}_{-w}(W_{vt,m;\text{in}}(\mathcal{C}_{x,\vec{v},\gamma})) &= \{y - wp \mid (y,p) \in W_{vt,m;\text{in}}(\mathcal{C}_{x,\vec{v},\gamma})\}\\ &= \{y + wp \mid (y,p) \in A_{vt}(\mathcal{C}_{x,\vec{v},\gamma}), p \in A_{-m}(\mathcal{C}_{\vec{v},\gamma})\}\\ &= \mathfrak{C}_w(W_{vt,-m;\text{out}})\end{aligned}$$

so that by Proposition 5.1

$$d(\mathfrak{C}_{-w}(\mathcal{W}_{vt,m;\text{in}}), \mathcal{A}^c_{\varepsilon(t+w)}) = (vt - mw) - \varepsilon(t+w) > (v - m - 2\varepsilon)w$$

which is greater than δw because $w < t$, $\varepsilon < \frac{v-m}{4}$, and $\delta < \frac{v-m}{2}$. Thus, we may apply Lemma A.5 to conclude that for any $\ell > 0$ there is some $C > 0$ such that

$$\|\chi_{\mathcal{A}^c_{\varepsilon(t+w)}}e^{iwH_0}P_\delta(\mathcal{W}_{vt,m;\text{in}})\|_{\text{op}} < Ct^{-\ell}$$

from which (5.9) follows immediately when combined with the Enss condition (3.1).

To prove (5.10), we fix $\psi \in \mathcal{H}$ compactly supported and choose R so that $\operatorname{supp}\psi \subset \mathcal{A}_0 + B_R$. Then

$$\|P_\delta(\mathcal{W}_{vt,m;\text{in}})e^{-iH_0t}\psi\| = \|P_\delta(\mathcal{W}_{vt,m;\text{in}})e^{-iH_0t}\chi_{\mathcal{A}_0+B_R}\psi\| \leq \|\chi_{\mathcal{A}_0+B_R}e^{iH_0t}P_\delta(\mathcal{W}_{vt,m;\text{in}})\|_{\text{op}}\|\psi\|$$

Again by Proposition 5.1

$$d(\mathfrak{C}_{-t}(\mathcal{W}_{vt,m;\mathrm{in}}),\mathcal{A}_0+B_R) > (v-m)t - R > \frac{v-m}{2}t > \delta t$$

for $t > \frac{2R}{v-m}$. Therefore, we can apply Lemma A.5, to get that

$$\|\chi_{\mathcal{A}_0+B_R} e^{iH_0 t} P_\delta(W_{vt,m;\mathrm{in}})\|_{\mathrm{op}} < C((v-m)t-R)^{-\ell}$$

from which it follows that

$$\lim_{t\to\infty} P_\delta(\mathcal{W}_{vt,m;\mathrm{in}})e^{-itH_0}\psi = 0$$

Density establishes (5.10), thus proving the lemma in full. □

Proof of Theorem 3.3. The key point is that in this case

$$\mathcal{W}_{n,m;\mathrm{out}} \cup \mathcal{W}_{n,m;\mathrm{in}} = \mathcal{A}_n \times \mathbb{R}^d \tag{5.11}$$

To see this, note that if $\mathcal{C}_i = \mathcal{C}_{x,\vec{v},\gamma}$ and $\gamma = \frac{\pi}{2}$ then $\mathcal{C}^c_{\vec{v},\gamma} \subset -\mathcal{C}_{\vec{v},\gamma}$ since if $y \in \mathcal{C}^c_{\vec{v},\gamma}$ we have

$$\langle y, \vec{v}\rangle \le \cos(\gamma)\|y\| = 0 \implies \langle -y, \vec{v}\rangle \ge 0 = \cos(\gamma)\|y\|$$

The inequality is strict up to a set of zero measure. In particular,

$$A^c_m(\mathcal{C}_{\vec{v},\gamma}) = \mathcal{C}^c_{\vec{v},\gamma} + \frac{m}{\sin(\gamma)}\vec{v} \subset -\mathcal{C}_{\vec{v},\gamma} + \frac{m}{\sin(\gamma)}\vec{v} = -A_{-m}(\mathcal{C}_{\vec{v},\gamma})$$

so that (5.11) holds.

Now, fix $\psi \in \mathcal{H}_{\mathrm{int}}$ and $v > 0$. Choose $m < v$ and δ sufficiently small and apply Proposition 5.9 to see that

$$\lim_{t\to\infty}\|P_\delta(\mathcal{A}_{vt}\times\mathbb{R}^d)\psi_t\| = \lim_{t\to\infty}\|P_\delta(\mathcal{W}_{vt,m;\mathrm{out}}\cup\mathcal{W}_{vt,m;\mathrm{in}})\psi_t\| = 0$$

Since

$$\|\chi_{\mathcal{A}_{\frac{vt}{2}}}\psi_t\| \le \|P_\delta(\mathcal{A}_{vt}\times\mathbb{R}^d)\psi_t\| + \|\chi_{\mathcal{A}_{\frac{vt}{2}}} P_\delta(\mathcal{A}^c_{vt}\times\mathbb{R}^d)\psi_t\|$$

and

$$\|P_\delta(\mathcal{A}_{vt}\times\mathbb{R}^d)\psi_t\| \le \|\chi_{\mathcal{A}_{2vt}}\psi_t\| + \|P_\delta(\mathcal{A}_{vt}\times\mathbb{R}^d)\chi_{\mathcal{A}^c_{2vt}}\psi_t\|$$

from Proposition A.4 we see that

$$\|\chi_{\mathcal{A}_{\frac{vt}{2}}}\psi_t\| + o(1) \le \|P_\delta(\mathcal{A}_{vt}\times\mathbb{R}^d)\psi_t\| \le \|\chi_{\mathcal{A}_{2vt}}\psi_t\| + o(1)$$

as $t\to\infty$. Therefore,

$$\mathcal{H}_{\mathrm{int}} \subset \{\psi\in\mathcal{H} \mid \forall v>0,\ \lim_{t\to 0}\|\chi_{\mathcal{A}_{vt}}\psi_t\| = 0\}$$

Conversely, if $\lim_{t\to\infty}\|\chi_{\mathcal{A}_{vt}}\psi_t\| = 0$ then by the above for any $\delta > 0$

$$\lim_{t\to\infty}\|P_\delta(\mathcal{A}_{vt}\times\mathbb{R}^d)\psi_t\| = 0$$

and thus for any $m > 0$

$$\|P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})\psi_t\|^2 = \langle P_\delta(\mathcal{W}_{vt,m;\mathrm{out}})^2\psi_t,\psi_t\rangle \le \langle P_\delta(\mathcal{A}_{vt}\times\mathbb{R}^d)\psi_t,\psi_t\rangle \xrightarrow{t\to\infty} 0$$

This proves the opposite inclusion

$$\{\psi\in\mathcal{H} \mid \forall v>0,\ \lim_{t\to 0}\|\chi_{\mathcal{A}_{vt}}\psi_t\| = 0\} \subset \mathcal{H}_{\mathrm{int}}$$

and allows us to conclude the equality of the two subspaces.

The same argument shows that

$$\mathcal{H}_{\mathrm{scat}} \subset \{\psi\in\mathcal{H} \mid \exists v>0 \lim_{t\to 0}\|\chi_{\mathcal{A}^c_{vt}}\psi_t\| = 0\}$$

because $\mathcal{A}^c_{vt} \times \mathbb{R}^d \subset \mathcal{W}^c_{vt,m;\text{out}}$ for any $m > 0$. Furthermore, if $\lim_{t\to\infty} \|\chi_{\mathcal{A}^c_{vt}}\psi_t\| = 0$ for some $v > 0$, then ψ is orthogonal to $\mathcal{H}_{\text{int}}$ since we have shown that any $\varphi \in \mathcal{H}_{\text{int}}$ must satisfy $\lim_{t\to\infty} \|\chi_{\mathcal{A}_{vt}}\varphi_t\| = 0$, for all $v > 0$. Therefore, $\psi \in \mathcal{H}^\perp_{\text{int}} = \overline{\mathcal{H}_{\text{scat}}}$, thus proving the opposite inclusion and concluding the proof. □

6. Examples

Example 2.1 (Single cone)**.** Suppose that $\{\mathcal{C}\}_{i\in\mathcal{I}}$ consists of a single cone $\mathcal{C} = \mathcal{C}_{x,\vec{v},\gamma}$. Then $\mathcal{D} = \overline{\mathcal{D}(\mathcal{C}_{\vec{v},\gamma})}$ and Theorem 3.2 gives the following microlocal description:

$$\Omega(\mathcal{D}) = \overline{\{\psi \in \mathcal{H} \mid \exists v, m, \delta_0 > 0, \text{ so that } \forall\delta \in (0, \delta_0) \lim_{t\to\infty} \|(P_\delta([A_{vt}(\mathcal{C}) \times A_m(\mathcal{C})]^c)\psi_t\| = 0\}}$$
$$\Omega(\mathcal{D})^\perp = \{\psi \in \mathcal{H} \mid \forall v, m > 0, \exists\delta_0 > 0 \text{ so that } \forall\delta \in (0, \delta_0) \lim_{t\to\infty} \|P_\delta(A_{vt}(\mathcal{C}) \times A_m(\mathcal{C}))\psi_t\| = 0\}$$

This indicates that $\Omega(\mathcal{D})$ consists of states which propagate into $\mathcal{C}$ with momenta in $A_m(\mathcal{C})$. When $\gamma < \frac{\pi}{2}$, this is the best description our theorems afford. It does not rule out a state in $\Omega(\mathcal{D})^\perp$ which propagates into $\mathcal{C}$, but with the wrong momenta and that thus could bounce off of the boundary of $\mathcal{C}$.

However, when $\gamma = \frac{\pi}{2}$, the potential is concentrated in a half-space and the Theorem 3.3 shows that in fact

$$\Omega(\mathcal{D}) = \overline{\{\psi \in \mathcal{H} \mid \exists v > 0, \lim_{t\to\infty} \|\chi_{A^c_{vt}(\mathcal{C})}\psi_t\| = 0\}}$$
$$\Omega(\mathcal{D})^\perp = \{\psi \in \mathcal{H} \mid \forall v > 0, \lim_{t\to\infty} \|\chi_{A_{vt}(\mathcal{C})}\psi_t\| = 0\}$$

because in this case we have shown that it is impossible for a state to propagate into $\mathcal{C}$ with momenta pointing away from $\mathcal{C}$ (this is the content of Proposition 5.9). Systems of this type, in particular of a vacuum coupled to a crystal (that is, a periodic potential) are physically important and have been studied, among elsewhere, in [7].

Example 2.2 (Non-convex cone)**.** Let $\mathcal{C} = \mathcal{C}_{x,\vec{v},\gamma}$ for $\gamma > \frac{\pi}{2}$. As explained in the introduction, we may choose $\{\mathcal{C}\}_{i\in\mathcal{I}}$ so that $\bigcup_{i\in\mathcal{I}} \mathcal{C}_i = \mathcal{C}$. It is easily verified that

$$A_r(\mathcal{C}) = \mathcal{A}_r$$

so the generalized Enss condition remains:

$$\|V\chi_{A_r(\mathcal{C})}\|_{\text{op}} = \|V\chi_{\mathcal{A}_r}\|_{\text{op}} \in L^1([0, \infty), dr)$$

Theorem 3.3 shows that in fact

$$\Omega(\mathcal{D}) = \overline{\{\psi \in \mathcal{H} \mid \exists v > 0, \lim_{t\to\infty} \|\chi_{A^c_{vt}(\mathcal{C})}\psi_t\| = 0\}}$$
$$\Omega(\mathcal{D})^\perp = \{\psi \in \mathcal{H} \mid \forall v > 0 \lim_{t\to\infty} \|\chi_{A_{vt}(\mathcal{C})}\psi_t\| = 0\}$$

as one would expect.

Example 2.3 (Short-range scattering)**.** As explained in the introduction, we may choose $\{\mathcal{C}\}_{i\in\mathcal{I}}$ so that $\mathcal{A}_r = B^c_r$. Relative to this collection of cones, the condition (3.1) becomes the classical Enns condition

$$\|V\chi_{B^c_r}\|_{\text{op}} \in L^1([0, \infty), dr)$$

which is one of many short-range scattering assumptions in the literature. Here, $\mathcal{D}$ is in fact equal to all of $\mathcal{H}$. In this setting, Theorem 3.3 shows that

$$\begin{aligned}\operatorname{Ran}(\Omega) &= \overline{\{\psi \in \mathcal{H} \mid \exists v > 0, \lim_{t\to\infty} \|\chi_{B_{vt}}\psi_t\| = 0\}}\\ \operatorname{Ran}(\Omega)^{\perp} &= \{\psi \in \mathcal{H} \mid \forall v > 0, \lim_{t\to\infty} \|\chi_{B_{vt}^c}\psi_t\| = 0\}\end{aligned}$$

This result may be contrasted with the usual asymptotic completeness statement for short-range scattering, which is

$$\begin{aligned}\operatorname{Ran}(\Omega) &= \mathcal{H}_{\mathrm{c}}(H)\\ \operatorname{Ran}(\Omega)^{\perp} &= \mathcal{H}_{\mathrm{pp}}(H)\end{aligned}$$

This latter description may be connected to the dynamics of H via the RAGE theorem [2, 14], which is a crucial ingredient in the original argument of Enss. A standard formulation of the RAGE theorem (see for example [17]) is

$$\begin{aligned}\mathcal{H}_{\mathrm{c}}(H) &= \{\psi \in \mathcal{H} \mid \lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T \|\chi_{B_n}\psi_t\|dt = 0\}\\ \mathcal{H}_{\mathrm{pp}}(H) &= \{\psi \in \mathcal{H} \mid \lim_{n\to\infty}\sup_{t\geq 0}\|\chi_{B_n^c}\psi_t\|dt = 0\}\end{aligned}$$

In particular, the space variable n is decoupled from t, whereas in order to get the spatial description, we fixed $n = vt$ for some velocity v.

Example 2.4 (Subspace potential)**.** Let S_r be the points within distance r from some fixed subspace of $\mathbb{R}^d$. We explained in the introduction that S_r may be written as $\mathcal{A}_r^c$ for some appropriately chosen collection of cones. In this setting, $\mathcal{D} = \mathcal{H}$ and Theorem 3.3 shows that if

$$\|V\chi_{S_r^c}\|_{\mathrm{op}} \in L^1([0,\infty), dr)$$

then

$$\begin{aligned}\operatorname{Ran}(\Omega) &= \overline{\{\psi \in \mathcal{H} \mid \exists v > 0, \lim_{t\to\infty} \|\chi_{S_{vt}}\psi_t\| = 0\}}\\ \operatorname{Ran}(\Omega)^{\perp} &= \{\psi \in \mathcal{H} \mid \forall v > 0, \lim_{t\to\infty} \|\chi_{S_{vt}^c}\psi_t\| = 0\}\end{aligned}$$

which recovers the main result of [3].

Example 2.5 (Broken subspace)**.** In this case, $\{\mathcal{C}\}_{i\in\mathcal{I}}$ consists of two cones $\mathcal{C}_1$ and $\mathcal{C}_2$, the first with $\gamma_1 < \frac{\pi}{2}$ and the second with $\gamma_2 = \pi - \gamma_1$. Let $\mathcal{D}_1 = \mathcal{D}(\mathcal{C}_1)$ and $\mathcal{D}_2 = \mathcal{D}(\mathcal{C}_2)$ be the domains of Ω corresponding to each cone. Then relative to $\mathcal{C}_1$ we obtain only a microlocal description

$$\begin{aligned}\Omega(\mathcal{D}_1) &= \overline{\{\psi \in \mathcal{H} \mid \exists v, m, \delta_0 > 0 \text{ so that } \forall \delta \in (0,\delta_0) \lim_{t\to\infty} \|(P_\delta([A_{vt}(\mathcal{C}_1) \times \mathcal{C}_1]^c)\psi_t\| = 0\}}\\ \Omega(\mathcal{D}_1)^{\perp} &= \{\psi \in \mathcal{H} \mid \forall v, m > 0, \exists \delta_0 > 0, \text{ so that } \forall \delta \in (0,\delta_0) \lim_{t\to\infty} \|P_\delta(A_{vt}(\mathcal{C}_1) \times A_m(\mathcal{C}_1))\psi_t\| = 0\}\end{aligned}$$

whereas for the second cone we obtain a purely spatial description, as seen in Example 2.2 above

$$\begin{aligned}\Omega(\mathcal{D}_2) &= \overline{\{\psi \in \mathcal{H} \mid \exists v > 0, \lim_{t\to\infty} \|\chi_{A_{vt}(\mathcal{C}_2)}\psi_t\| = 0\}}\\ \Omega(\mathcal{D}_2)^{\perp} &= \{\psi \in \mathcal{H} \mid \forall v > 0, \lim_{t\to\infty} \|\chi_{A_{vt}^c(\mathcal{C}_2)}\psi_t\| = 0\}\end{aligned}$$

In other words, any state which propagates into the larger cone $\mathcal{C}_2$ at a linear rate must be a scattering state, but for $\mathcal{C}_1$ this is only the case for states with momenta which also point into $\mathcal{C}_1$.

Appendix A. Existence of the POVM P_δ

Proposition A.1. *There exists a Positive Operator Valued Measure (POVM), P_δ, defined on the phase space $\mathbb{R}^d_x \times \mathbb{R}^d_p$, with the following properties, for any $E \subset \mathbb{R}^{2d}$ Borel*

(1) *(Observable)* $P_\delta(\mathbb{R}^{2d}) = \mathrm{id}$.

(2) *(Momentum localization) Let $B \subset \mathbb{R}^d$ and $D \subset \mathbb{R}^d$ be Borel sets such that $d(B, D) > \delta$. Then for any $E \subset \mathbb{R}^d \times B$ Borel and $\psi \in \mathcal{H}$ such that* $\operatorname{supp} \hat{\psi} \subset D$

$$P_\delta(E)\psi = 0$$

(3) *(Approximate space localization) Let $A \subset \mathbb{R}^d$ and $D \subset \mathbb{R}^d$ be Borel sets so that $d(D, A) > 0$. Then for any $\ell > 0$ there exists some constant $C > 0$ depending only on η_δ so that for all $E \subset A \times \mathbb{R}^d$*

$$\|P_\delta(E)\chi_D\|_{op} < C[d(A, D)]^{-\ell}$$

(4) *(Microlocal non-stationary phase estimate) Let $\mathfrak{C}_t(E) \subset \mathbb{R}^d$ denote the classically allowed region associated to $E \subset \mathbb{R}^{2d}$ at time t:*

$$\mathfrak{C}_t(E) = \{x + tp \mid (x, p) \in E\}$$

Let $F \subset \mathbb{R}^d$ be Borel. For any $\ell > 0$ there exists $C > 0$ such that

$$\|\chi_F e^{-itH_0} P_\delta(E)\|_{\mathrm{op}} \leq C d(|t|)^{-\ell}$$

for all t such that $d(t) := d(\mathfrak{C}_t(E), F) > \delta|t|$.

(5) *(Spatial non-stationary phase estimate) Let $\{A_t\}_{t\geq 0}$ be collections of Borel subsets of $\mathbb{R}^d$.*
Then for any $\varphi \in \mathcal{S}$ such that $\operatorname{supp} \hat{\varphi} \Subset D$ Borel, $\ell > 0$, and $\varepsilon > 0$ there exists some constant $C(\psi, \ell, \varepsilon, \delta) > 0$ such that

$$\|P_\delta(A_t \times \mathbb{R}^d) e^{-itH_0}\varphi\| < Ct^{-\ell}$$

for all t such that $d(A_t, tD) > \varepsilon t$.

Proof. To this end, we will use the phase space observable formalism developed in [4, 6] and used in [3].

Choose $\eta \in \mathcal{S}$, such that $\|\eta\| = 1$ and $\operatorname{supp} \hat{\eta} \subset B_1$. Let η_δ be such that $\hat{\eta}_\delta(p) = \delta^{-\frac{d}{2}}\hat{\eta}(\frac{p}{\delta})$, a rescaling of η, so that $\operatorname{supp} \hat{\eta}_\delta \subset B_\delta$ and $\|\eta_\delta\| = 1$.

Now define the following family of coherent states by translating η_δ in phase space:

$$\hat{\eta}_{x,p;\delta}(\xi) = e^{-ix\xi}\hat{\eta}_\delta(\xi - p)$$

We use this to define a family, depending on $\delta > 0$, of positive-operator-valued measures: for any $E \subset \mathbb{R}^{2d}$ Borel and $\psi \in \mathcal{H}$ let

$$P_\delta(E)\psi = (2\pi)^{-d} \iint_E \langle \eta_{x,p;\delta}, \psi\rangle \, \eta_{x,p;\delta} \, dx \, dp$$

The various properties of P_δ are proved in a series of propositions below.

In Appendix A of [3] we proved the following properties of P_δ:

Proposition A.2 (Observable)**.** *For any $\delta > 0$ we have $P_\delta(\mathbb{R}^{2d}) = \mathrm{id}$.*

Proposition A.3 (Momentum localization)**.** *Let $B \subset \mathbb{R}^d$ and $D \subset \mathbb{R}^d$ be Borel sets such that $d(B, D) > \delta$. Then for any $E \subset \mathbb{R}^d \times B$ Borel and $\psi \in \mathcal{H}$ such that* $\operatorname{supp} \hat{\psi} \subset D$

$$P_\delta(E)\psi = 0$$

Proposition A.4 (Approximate space localization). *Let $A \subset \mathbb{R}^d$ Borel and any set $D \subset \mathbb{R}^d$ Borel such that $d(D, A) > 0$, for any $\ell > 0$ we have some constant $C > 0$ depending only on η_δ so that*

$$\|P_\delta(A \times \mathbb{R}^d)\chi_D\|_{\mathrm{op}} < C[d(A, D)]^{-\ell}$$

Finally we prove two estimates relating P_δ to the free propagator e^{-itH_0}, both based on the principle of of non-stationary phase. The first is similar to Lemma 2 of Theorem XI.112 in [13], but adapted to P_δ. This lemma and its proof are similar to Lemma 3 in [22].

Lemma A.5 (Microlocal non-stationary phase estimate). *Let $\mathfrak{C}_t(E) \subset \mathbb{R}^d$ denote the classically allowed region associated to $E \subset \mathbb{R}^{2d}$ at time t:*

$$\mathfrak{C}_t(E) = \{x + tp \mid (x, p) \in E\}$$

Let $F \subset \mathbb{R}^d$ be Borel. For any $\ell > 0$ there exists $C > 0$ such that

$$\|\chi_F e^{-itH_0} P_\delta(E)\|_{\mathrm{op}} \le C d(|t|)^{-\ell}$$

for all t such that $d(t) := d(\mathfrak{C}_t(E), F) > \delta|t|$.

Proof. We start by noting that for any $\psi \in \mathcal{H}$, by the boundedness of P_δ

$$\|P(E)\psi\|^2 = \langle \psi, P^2(E)\psi\rangle \le \langle \psi, P(E)\psi\rangle = (2\pi)^{-d} \iint\limits_E \langle \eta, \psi\rangle \langle \psi, \eta\rangle \, dxdp = (2\pi)^{-d} \iint\limits_E |\langle \eta, \psi\rangle|^2 dxdp$$

We will estimate the norm of the adjoint operator $P_\delta(E)e^{itH_0}\chi_F$. For $\psi \in \mathcal{H}$, by the above inequality

$$\begin{aligned}
\|P_\delta(E)e^{itH_0}\chi_F\psi\|^2 &\le (2\pi)^{-d} \iint\limits_E |\left\langle \eta_{x,p;\delta}, e^{itH_0}\chi_F\psi\right\rangle|^2 \, dx \, dp \\
&= (2\pi)^{-d} \iint\limits_E |\int\limits_{\mathbb{R}^d} \overline{e^{-itH_0}\eta_{x,p;\delta}}(y)\chi_F(y)\psi(y) \, dy|^2 \, dx \, dp
\end{aligned}$$

We now compute

$$\begin{aligned}
&(e^{-itH_0}\eta_{x,p;\delta})(y) = (2\pi)^{-\frac{d}{2}} \int\limits_{\mathbb{R}^d} e^{i\xi\cdot(y-x) - it\frac{\xi^2}{2}} \hat{\eta}_\delta(\xi - p) \, d\xi \\
&= e^{ip\cdot(y-x) - it\frac{p^2}{2}} (2\pi)^{-\frac{d}{2}} \int\limits_{\mathbb{R}^d} e^{i\xi\cdot(y-x) - it\frac{\xi^2}{2} - it\xi\cdot p} \hat{\eta}_\delta(\xi) \, d\xi = e^{ip\cdot(y-x) - it\frac{p^2}{2}} (e^{-itH_0}\eta_\delta)(y - (x + tp))
\end{aligned}$$

Recalling that for any $(x, p) \in E$ and $y \in F$, $|y - (x + tp)| > d(t)$, we see that

$$\begin{aligned}
&|\int\limits_{\mathbb{R}^d} \overline{e^{-itH_0}\eta_{x,p;\delta}}(y)\chi_F(y)\psi(y) \, dy| = |\int\limits_{\mathbb{R}^d} e^{-ipy}\overline{(e^{-itH_0}\eta_\delta)}(y - (x + tp))\chi_F(y)\psi(y) \, dy| \\
&= |\int\limits_{\mathbb{R}^d} e^{-ipy}\overline{(e^{-itH_0}\eta_\delta)}(y - (x + tp))\chi_{\{|y-(x+tp)| > d(t)\}}(y)\chi_F(y)\psi(y) \, dy| \\
&= (2\pi)^{\frac{d}{2}} |\mathcal{F}[\overline{(e^{-itH_0}\eta_\delta)}(\cdot - (x + tp))\chi_{\{|\cdot-(x+tp)| > d(t)\}}(\cdot)\chi_F(\cdot)\psi(\cdot)](p)|
\end{aligned}$$

We now perform the change of variables $x' = x + tp$ and apply the Plancharel theorem to see that

$$\begin{aligned}
&(2\pi)^{-d} \iint\limits_{E} |\int\limits_{\mathbb{R}^d} \overline{e^{-itH_0}\eta_{x,p;\delta}}(y)\chi_F(y)\psi(y)\,dy|^2\,dx\,dp \\
&\leq (2\pi)^{-d} \iint\limits_{\mathbb{R}^{2d}} (2\pi)^d |\mathcal{F}[\overline{(e^{-itH_0}\eta_\delta)}(\cdot - (x+tp))\chi_{\{|\cdot-(x+tp)|>d(t)\}}(\cdot)\chi_F(\cdot)\psi(\cdot)](p)|^2\,dx\,dp \\
&=_{x'=x+tp} \iint\limits_{\mathbb{R}^{2d}} |\mathcal{F}[\overline{(e^{-itH_0}\eta_\delta)}(\cdot - x')\chi_{\{|\cdot-x'|>d(t)\}}(\cdot)\chi_F(\cdot)\psi(\cdot)](p)|^2\,dx'\,dp \\
&= \iint\limits_{\mathbb{R}^{2d}} |\overline{(e^{-itH_0}\eta_\delta)}(y-x')\chi_{\{|y-x'|>d(t)\}}(y)\chi_F(y)\psi(y)|^2\,dx'\,dy \\
&= \int\limits_{\mathbb{R}^d} |\chi_F(y)\psi(y)|^2 \int\limits_{\{x' | |y-x'|>d(t)\}} |\overline{(e^{-itH_0}\eta_\delta)}(y-x')|^2\,dx'\,dy \leq \|\psi\|^2 \int\limits_{B^c_{d(t)}} |(e^{-itH_0}\eta_\delta)(x')|^2\,dx'
\end{aligned}$$

Since $d(t) > \delta t$ by assumption and $\operatorname{supp}\hat{\eta} \subset B_\delta$, we see that if $x' \in B^c_{d(t)}$ then $\frac{x'}{t} \notin B_\delta$ so we may apply Lemma 4.2 to see that for any $\ell > 0$ there exists $C > 0$ depending only on η and δ such that

$$\int\limits_{\{x' | |y-x'|>d(t)\}} |\overline{(e^{-itH_0}\eta_\delta)}(y-x')|^2\,dx' \leq C \int\limits_{B^c_{d(t)}} (1 + \|x'\| + |t|)^{-\ell}\,dx' \leq C(1 + d(t) + |t|)^{-\ell+d}$$

Thus, we conclude that

$$\|P_\delta(E)e^{itH_0}\chi_F\psi\| \leq C(1 + |t| + d(t))^{-\ell+d}\|\psi\|^2$$

as claimed. □

The second lemma is essentially a standard non-stationary phase estimate on e^{-itH_0}: see, for instance, the Corollary to Theorem XI.14 from [13].

Proposition A.6 (Spatial non-stationary phase estimate)**.** *Let $\{A_t\}_{t\geq 0}$ be a collection of Borel subsets of $\mathbb{R}^d$. Then for any $\varphi \in \mathcal{S}$ such that $\operatorname{supp}\hat{\varphi} \Subset D$ Borel, $\ell > 0$, and $\varepsilon > 0$ there exists some constant $C(\psi, \ell, \varepsilon, \delta) > 0$ such that*

$$\|P_\delta(A_t \times \mathbb{R}^d)e^{-itH_0}\varphi\| < Ct^{-\ell}$$

for all t such that $d(A_t, tD) > \varepsilon t$.

Proof. Let $\varphi \in \mathcal{S}$ and $\ell, \varepsilon > 0$ and D be as above. Then we can write

$$\|P_\delta(A_t \times \mathbb{R}^d)e^{-itH_0}\varphi\| \leq \|P_\delta(A_t \times \mathbb{R}^d)\chi_{[A_t + B_{\frac{\varepsilon}{2}t}]^c}\|_{\text{op}}\|\varphi\| + \|\chi_{A_t + B_{\frac{\varepsilon}{2}t}}e^{-itH_0}\varphi\|$$

Since $d(A_t, [A_t + B_{\frac{\varepsilon}{2}t}]^c) > \frac{\varepsilon}{2}t$, by Property A.4 we get that

$$\|P_\delta(A_t \times \mathbb{R}^d)\chi_{[A_t + B_{\frac{\varepsilon}{2}t}]^c}\|_{\text{op}} < Ct^{-\ell}$$

We can write

$$\|\chi_{A_t + B_{\frac{\varepsilon}{2}}}e^{-itH_0}\varphi\|^2 = \int\limits_{A_t + B_{\frac{\varepsilon}{2}}} |e^{-itH_0}\varphi(y)|^2 dy$$

Next, we note that

$$d(A_t + B_{\frac{\varepsilon}{2}t}, tD) \geq d(A_t, tD) - \frac{\varepsilon}{2}t > \frac{\varepsilon}{2}t$$

So we conclude that $y \in A_t + B_{\frac{\varepsilon}{2}t}$ implies that $\frac{y}{t} \notin D$, and so by Lemma 4.3 we get

$$|e^{-itH_0}\varphi(y)| \leq C(1 + \|y\| + t)^{-\ell-d}$$

Therefore,

$$\|\chi_{A_t+B_{\frac{\varepsilon}{2}t}} e^{-itH_0}\varphi\|^2 \leq (1+t)^{-\ell}$$

as needed. □

With this lemma, we have proved all the claimed properties of P_δ. □

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