

$L^1 \rightarrow L^\infty$ DISPERSIVE ESTIMATES FOR COULOMB WAVES

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ABSTRACT. We show the time decay of spherically symmetric Coulomb waves in \mathbb{R}^3 for the case of a repulsive charge. By means of a distorted Fourier transform adapted to $H = -\Delta + q \cdot |x|^{-1}$, with $q > 0$, we explicitly compute the kernel of the evolution operator e^{itH} . A detailed analysis of the kernel is then used to prove that for large times, e^{itH} obeys an $L^1 \rightarrow L^\infty$ dispersive estimate with the natural decay rate $t^{-\frac{3}{2}}$.

1. INTRODUCTION

1.1. Motivation and main result. The time-dependent *Coulomb wave equation* describes the quantum dynamics of a charge under the influence of a long-range spherically symmetric potential in \mathbb{R}^3 :

$$(1.1) \quad \begin{aligned} (i\partial_t - H)u &= 0, & H &:= -\Delta + \frac{q}{|x|}, & q &\in \mathbb{R} \\ u(0, x) &= f(x) \end{aligned}$$

In the attractive case, $q < 0$, this equation has been used since the birth of quantum mechanics to describe the time-evolution of the electron in a hydrogen atom [39]. In this paper, we focus on the repulsive case, $q > 0$, which was introduced by Yost, Wheeler and Breit in 1936 [45] in connection with the interaction of charges of the same sign. The corresponding solutions are called *Coulomb waves* [13].

As is well known, the free Schrödinger equation in \mathbb{R}^3 (that is (1.1) without the potential term) enjoys many dispersive estimates describing the spreading of the wave packet, the most fundamental of which is

$$(1.2) \quad \|e^{it\Delta} f\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C}{|t|^{3/2}} \|f\|_{L^1(\mathbb{R}^3)},$$

with constant independent of f and t . This is a direct consequence of the fact that the propagator of the free Schrödinger equation in \mathbb{R}^3 can be computed as $e^{it\Delta} f = k_t * f$, where

$$k_t(x) = (4\pi it)^{-3/2} e^{\frac{ix|^2}{4it}}.$$

Our goal in this paper is to extend the estimate (1.2) to e^{itH} for radial data. Namely, we prove the following theorem:

Theorem 1.1. *Let $H = -\Delta + \frac{q}{|x|}$ be the Coulomb Hamiltonian in \mathbb{R}^3 , with $q > 0$. Then, for any spherically symmetric function f one has the following a priori estimate*

$$(1.3) \quad \|e^{itH} f\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C}{|t|^{3/2}} \|f\|_{L^1(\mathbb{R}^3)}, \quad |t| \geq 1.$$

with constant independent of f and t .

As explained in the appendix, H is a self-adjoint operator on $H^2(\mathbb{R}^3)$ so the meaning of e^{itH} is clear. In later work, we plan to extend this result to non-radial functions, see the discussion in Section 1.3.

Besides its intrinsic interest, the estimate (1.3) serves as an initial step towards understanding the modified scattering of the *Hartree* or *Gross-Pitaevskii* equation in \mathbb{R}^3 :

$$(i\partial_t - \Delta)u + ((-\Delta)^{-1}|u|^2)u = 0.$$

This is a nonlinear equation modeling the dynamics of non-relativistic bosonic many-body particle systems in the mean-field limit, which has been extensively studied together with the NLS equation, see e.g. [20, 24]. Indeed, taking radial perturbations leads to the equation

$$\left(i\partial_t - \partial_r^2 + \frac{1}{r}\right)v + \varepsilon^2 v \int_{\mathbb{R}_+} \frac{1}{\max\{r, s\}} |v(s, t)|^2 ds = 0,$$

with $0 < \varepsilon \ll 1$ whose linearization is given by the Coulomb equation (1.1) for radial data. Thus, any investigation of the nonlinear scattering phenomenon of the Hartree equation must begin with a detailed understanding of the long-time asymptotics of solutions to (1.1).

1.2. Prior work. The problem of extending pointwise dispersive estimates to Hamiltonians of the form $-\Delta + V(x)$ has attracted considerable attention, given that these estimates serve as crucial tools for the subsequent analysis of both linear and nonlinear problems. In the linear setting, they give rise to Strichartz estimates and in the nonlinear realm they can be used to prove the stability of solitons, see e.g. [37, 38, 40, 42]. Nevertheless, most of the existing studies rely on either the perturbation of the free resolvent operator or the use of Duhamel's formula. Consequently, these approaches require that the potential V be bounded or small in some suitable sense or decay faster than $|x|^{-1}$ at infinity, see the notable references [3, 14, 19, 23, 36, 44] for some of the diverse methods employed in this area. From this perspective, the Coulomb potential $|x|^{-1}$ is pathological because of its singularity at the origin and slow decay at infinity. For instance, while pointwise estimates have been investigated for inverse square potentials [16, 26], the same methods cannot be directly applied to H due to the differing scaling behavior between the Coulomb potential and the Laplacian.

To the best of our knowledge, there is only one study quantifying the dispersive properties of the Hamiltonian H . In [29], Mizutani considers the more general operator

$$H_1 := -\Delta + Z|x|^{-\mu} + \varepsilon V_S(x)$$

on \mathbb{R}^n where $\mu \in (0, 2)$ and $|\partial_x^\alpha \{V_S(x)\}| \leq C\langle x \rangle^{-1-\mu-|\alpha|}$, which for $\varepsilon = 0$ and $\mu = 1$ is our operator H . For $\varepsilon \geq 0$ sufficiently small depending on Z, μ , and V_S , they show Strichartz estimates (including the endpoint) for H_1 , i.e.,

$$\|e^{itH_1} f\|_{L^p(\mathbb{R}; L^q(\mathbb{R}^n))} \leq C\|f\|_{L^2(\mathbb{R}^n)}, \quad \frac{2}{p} = \frac{n}{2} - \frac{n}{q}, \quad (n, p, q) \neq (2, \infty, \infty).$$

We emphasize that our estimates are instead pointwise in that they control the L^∞ norm for all large t . Furthermore, we explicitly compute the kernel of the evolution operator e^{itH} for radial data, which may be of independent interest.

1.3. Overview of the proof. Though we will mostly focus in radial waves, let us consider the spherical decomposition of $L^2(\mathbb{R}^3) = \bigoplus_{\ell=0}^\infty L^2(\mathbb{R}^+, r^2 dr) \otimes L_\ell$, where $r := |x|$ and L_ℓ denotes the ℓ -th eigenspace of spherical harmonics of angular momentum $\ell = 0, 1, 2, \dots$. The Coulomb Hamiltonian H restricted to L_ℓ is then unitarily equivalent to the Sturm-Liouville operator $-\partial_r^2 + \frac{\ell(\ell+1)}{r^2} + \frac{q}{r}$:

$$H|_{L_\ell} = H_{\ell, q} := r^{-1} \left(-\partial_r^2 + \frac{\ell(\ell+1)}{r^2} + \frac{q}{r} \right) r.$$

Thus, any $f \in L^2(\mathbb{R}^3)$ can be represented as

$$f(r, \omega) = r \sum_{\ell=0}^\infty f_\ell(r) Y_{\ell, m}(\omega), \quad f_\ell(r) := \sum_{m=-\ell}^\ell \langle f(r, \cdot), Y_{\ell, m} \rangle, \quad r \in \mathbb{R}^+, \omega \in \mathbb{S}^2,$$

where the inner product is understood in the sense of $L^2(\mathbb{S}^2)$, $f_\ell \in L^2(\mathbb{R}^+, r^2 dr)$ and $Y_{\ell, m}$ denotes the (ℓ, m) -spherical harmonic, i.e. $-\Delta_{\mathbb{S}^2} Y_{\ell, m} = \ell(\ell+1) Y_{\ell, m}$.

In this paper, we treat the radial sector, $\ell = 0$, leaving the analysis of other angular momenta to a subsequent work. As such, we must understand the half-line Schrödinger operator

$$-\frac{d^2}{dr^2} + \frac{q}{r}.$$

This operator is not essentially self-adjoint (indeed, it is limit circle at $r = 0$), so we must be careful to choose the correct self-adjoint extension whose dynamics coincide with that of $H_{0, q}$, which we denote \mathcal{L}_q . We refer the reader to the appendix for the construction of \mathcal{L}_q and its relevant properties.

To explicitly describe the time-evolution of \mathcal{L}_q , we derive its *distorted Fourier transform*, which diagonalizes the operator. This consists of a distorted Fourier basis of appropriately selected

generalized eigenfunctions of \mathcal{L}_q and a spectral measure $\rho(d\sigma)$, which yields the representation

$$(1.4) \quad e^{it\mathcal{L}_q}g(r) = \int_0^\infty \int_0^\infty e^{it\sigma^2} \phi_q(\sigma, r) \phi_q(\sigma, s) g(s) ds \rho(d\sigma).$$

While the existence of such a transform is quite classical for many classes of potentials, the potential r^{-1} is *strongly singular*, so that additional care is required to develop its spectral theory. The distorted Fourier transform of strongly singular potentials has been studied in [18]. However, the Coulomb potential is not singular enough to apply the results therein verbatim. In the appendix, we adapt the results of [18] to treat the Coulomb case, and derive a distorted Fourier basis and spectral measure given by

$$(1.5) \quad \phi_q(\sigma, r) = (2i\sigma)^{-1} M_{\frac{iq}{2\sigma}, \frac{1}{2}}(2i\sigma r), \quad d\rho(\sigma) = 2\mu^2(\sigma) d\sigma$$

where $\mu^2(\sigma) = q\sigma[e^{\frac{q\pi}{\sigma}} - 1]$, $\mu \geq 0$ and $M_{\frac{iq}{2\sigma}, \frac{1}{2}}(2i\sigma r)$ is Whittaker-M function, see 13.14 in [8]. We also mention work of Fulton [17], in which Frobenius solutions from zero are used to derive the distorted Fourier transform and spectral measure in the case of strongly singular potentials.

Substituting (1.5) into (1.4) and employing the relation $H_{0,q} = r^{-1}\mathcal{L}_q r$, we obtain

$$(1.6) \quad \begin{aligned} e^{itH_{0,q}}g(r) &= \frac{2q}{r} \int_0^\infty \int_0^\infty e^{itq^2\sigma^2} e(q\sigma, r) e(q\sigma, s) s g(s) ds d\sigma \\ &= \int_0^\infty K_t(r, s) s^2 g(s) ds, \end{aligned}$$

where

$$(1.7) \quad e(q\sigma, r) = \mu(q\sigma) \phi_1(q\sigma, r), \quad K_t(r, s) = \frac{2q}{rs} \int_0^\infty e^{itq^2\sigma^2} e(q\sigma, r) e(q\sigma, s) d\sigma.$$

Therefore, Theorem 1.1 holds provided that we establish

$$(1.8) \quad \sup_{r \geq s > 0} |K_t(r, s)| \lesssim t^{-3/2}, \quad t \geq 1.$$

It should be stressed that, despite the apparent simplicity of K_t in terms of special functions, we require several very delicate approximations in order to prove (1.8). In particular, for use in the oscillatory integral defining K_t , it is important that we obtain C^2 control jointly in the variables r and σ . While for large σ known integral representation of $e(q\sigma, r)$ suffice, as $\sigma \rightarrow 0$ these turn out to be useless. Therefore, one needs to perform a detailed analysis of the eigenvalue problem

$$(1.9) \quad -\frac{d^2 f}{dr^2} + \frac{f}{r} = \sigma^2 f$$

as $\sigma \rightarrow 0$, in which we have conveniently normalized the charge constant to $q = 1$. By rescaling, this loses no generality so we adopt this convention from now on. The fundamental set of solutions to (1.9) consists of $M_{\frac{i}{2\sigma}, \frac{1}{2}}(2i\sigma r)$ and $W_{\frac{i}{2\sigma}, \frac{1}{2}}(2i\sigma r)$, the latter being the Whittaker- W function. With $\phi(\sigma, r) := \phi_1(\sigma, r)$, as r approaches 0, for any fixed $\sigma > 0$

$$(1.10) \quad \phi(\sigma, r) = r(1 + O(\sigma r))$$

whereas as for $r \rightarrow \infty$, by equations (13.14.32), (13.14.21), and (5.4.3) in [8], we have that

$$(1.11) \quad \phi(\sigma, r) \sim C_0 \sigma^{-\frac{1}{2}} [e^{\frac{\pi}{\sigma}} - 1]^{\frac{1}{2}} \sin(\Theta(\sigma, r))$$

$$(1.12) \quad \Theta(\sigma, r) = \sigma r - \frac{\log(2\sigma r)}{2\sigma} + \theta_0(\sigma)$$

for some absolute constant C_0 and phase correction $\theta_0(\sigma)$.

Heuristically, one may see these asymptotics via the WKB method. Let

$$(1.13) \quad \rho_\sigma(s, r) = \int_s^r \sqrt{|Q|} du, \quad Q = u^{-1} - \sigma^2$$

be the *Agmon distance* [2, Ch.5] from s to r . To the right of the *turning point* $r_* = \sigma^{-2}$, WKB predicts that ϕ will grow as $e^{\rho_\sigma(0, r_*)}$ before oscillations set in, which are governed by $e^{i\rho_\sigma(r_*, r)}$. Since

$$\rho_\sigma(0, r_*) = \frac{\pi}{2\sigma}$$

and

$$\rho_\sigma(r_*, r) = \sigma^{-1} \left(r\sigma^2 + \frac{1}{2} \log(r\sigma^2) + \frac{1}{2} - \log(2) + O(r\sigma^2) \right),$$

the predictions of WKB exactly match the asymptotics (1.11), where $\sigma^{-\frac{1}{2}}$ may be regarded as the usual WKB prefactor of $Q^{-\frac{1}{4}}$. Precisely within equation (1.7), the role of the μ multiplier is adjusting the distorted Fourier basis in a manner that leads to

$$\lim_{r \rightarrow \infty} |e(\sigma, r) - c_0 \sin(\Theta(\sigma, r))| = 0$$

for all $\sigma > 0$ and an absolute constant $c_0 \in \mathbb{R}$.

To make the approximation precise, we use the *Liouville-Green*(LG) transform [32, Ch. 6], which is standard practice for semi-classical problems with a simple turning point. Indeed, similar semi-classical problems with inverse square or exponentially decaying potentials have been studied in [6, 7, 9, 38] (see Section 2 for a discussion on the differences). The LG transform is a change of variables ζ that transforms a second order ODE with a simple turning point into a *perturbed Airy equation* of the form

$$-\sigma^2 \frac{d^2 f}{d\zeta^2} = (\zeta + \sigma^2 V(\zeta)) f$$

for a suitable potential V . In this way, we obtain fundamental systems of solutions to (1.9) with good asymptotics on both sides of the turning point, see Proposition 2.7 and Proposition 2.11. Unfortunately, the approximation on the left side of the turning point cannot be extended all the way to the origin due to the local singularity in $V(\zeta)$ introduced by the Coulomb potential. However, referring to Nakamura's investigation in [31], it is reasonable to anticipate a rapid decay of $e(\sigma, r)$ as σ approaches 0. In [31], the resolvent operator of $-\Delta + W$ where $|W(x)| \lesssim \langle x \rangle^{-\rho}$ for $\rho \in (0, 2)$ is studied. It was proved that if W is also positive then there exist $\beta, \gamma > 0$, such that

$$(1.14) \quad \|F(|x| \leq \beta\sigma^2)F((-\Delta + W) \leq \sigma^2)\|_{L^2 \rightarrow L^2} \lesssim \exp(-\gamma\sigma^{\frac{2}{\rho}-1}), \quad \sigma^2 \in (0, 1]$$

where $F(A)$ denotes the characteristic function of the set A . It is important to note that the Coulomb potential takes the form of W with $\rho = 1$ as x becomes large. Therefore, we should compare our estimate on $e(\sigma, r)$ with $e^{-\frac{\gamma}{\sigma}}$ as $\sigma \rightarrow 0$. With this consideration, we employ a different transformation of (1.9) around zero and derive an approximation to $e(\sigma, r)$ via modified Bessel functions [8, Ch.10] that captures the expected behavior and is unable to be extended up to the turning point. We then connect this approximation to the Airy approximation mentioned above. Lemma 2.4 below provides a complementary perspective to the estimate given in (1.14).

It appears to us that one cannot avoid two connection problems when considering potentials that decay like r^{-1} . Indeed, in [33], the same problem arose while performing a turning point analysis of a similar ODE arising in the context of the Klein-Gordon equation on a stationary spherically symmetric black hole. The approximations in terms of Airy and Bessel functions obtained in Section 5 of that work are similar to those we obtain in Section 2.

We also mention that while approximations of Coulomb eigenfunctions via Bessel and Airy functions have appeared in the literature since the 50's [1, 12, 13], the form of these approximations are completely unsuitable for use inside the oscillatory integral (1.6) as we require explicit estimates of C^2 -errors in both r and the semi-classical parameter σ . To the best of our knowledge, the approximations derived in Sections 2 and 3 are novel. In particular, the results in the aforementioned references cannot be applied directly in the proof of estimate (1.8), for which one needs the control of several derivatives in order to extract the time decay from the phase. See Section 4 for the oscillatory estimates that lead to the proof of (1.8).

1.4. Notation and conventions. For the benefit of the reader, we define here the notation and conventions we use throughout this paper:

- $\langle x \rangle = (1 + x^2)^{\frac{1}{2}}$.

- $f(x) \lesssim g(x)$ indicates that there exists some constant $C > 0$ so $f(x) \leq Cg(x)$ for all x in some specified domain. We will also use this notation for functions that depend on several variables.
- $f(x) \sim g(x)$ indicates that $cg(x) \leq f(x) \leq Cg(x)$ for some $C > c > 0$ independent of x .
- $a(x, \sigma) = O_k(\sigma^m x^p)$ indicates that $|\partial_\sigma^j \{a(x, \sigma)\}| \lesssim \sigma^{m-j} x^p$ for $j = 0, 1, \dots, k$.
- $\chi_c(x) : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a smooth cut-off function supported on $[0, c]$ that is equal to 1 when $x \leq \frac{2c}{3}$, and $\tilde{\chi}_c(x) = 1 - \chi_c(x)$.
- For two functions $f(x)$ and $g(x)$, $W[f, g](x)$ denotes their Wronskian

$$W[f, g](x) = f(x)g'(x) - f'(x)g(x)$$

1.5. Organization of the paper. The paper is organized as follows. In Sections 2 and 3, we derive approximations to the Whittaker M-function $e(\sigma, r)$ for small and large energies σ , respectively. We devote Section 4 to the proof of our main Theorem 1.1. For this purpose, we employ the previous eigenfunction approximations to estimate the oscillatory integral (1.8), which leads immediately to the estimate (1.3). Finally, in an Appendix, we provide the details on the construction of the distorted Fourier basis (1.7) and thus demonstrate the form of the kernel K_t .

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2. EIGENFUNCTION APPROXIMATION: SMALL ENERGIES

The main aim of this section is to find a good approximation for the distorted Fourier basis $e(\sigma, r) = -i\sigma^{-\frac{1}{2}}[e^{\frac{\pi}{\sigma}} - 1]^{-\frac{1}{2}}M_{\frac{i}{2\sigma}, \frac{1}{2}}(2i\sigma r)$ when $\sigma < c$ for some sufficiently small c . The principal findings from this section that we employ for the analysis of the oscillatory integrals are summarized in Proposition 2.13, Corollary 2.14, and Corollary 2.17.

In the following section, we consider $q = 1$. In particular, we construct approximate solutions to the ODE

$$(2.1) \quad -f''(r) + \frac{f(r)}{r} = \sigma^2 f(r)$$

for σ small.

It is convenient to change to the variable $x = \sigma^2 r$ whereupon (2.1) becomes

$$(2.2) \quad -\sigma^2 f''(x) + (x^{-1} - 1)f(x) = 0.$$

As $\sigma \rightarrow 0$, (2.2) is a semi-classical problem with a simple turning point at $x = 1$. Thus, we use the Liouville-Green transform to obtain solutions in terms of Airy functions, which is standard practice for such ODEs (see Chapter 11 in [32]). In this section, we closely follow [7] where a similar analysis is performed for an ODE with a potential with r^{-2} asymptotics. Unlike in [7], we cannot extend the Airy function approximation to a neighborhood of $x = 0$. Indeed, the error in the approximation blows up in σ as $x \rightarrow 0$ due to the exponential growth of $\text{Bi}(z)$ as $z \rightarrow \infty$, see Proposition 2.7. Therefore, in this regime we resort to an approximation in terms of the modified Bessel function I_1 . Note that Erdélyi and Swanson use I_1 in [13] to approximate the Whittaker function $M_{\frac{i}{2\sigma}, \frac{1}{2}}(2i\sigma r)$, but the error term derived there is not suitable for our purposes. Therefore, our analysis in the next section may be regarded as a refined version of theirs.

2.1. Bessel function approximation: $x \ll 1$. In this section, we approximate $e(\sigma, r)$ in terms of the modified Bessel function I_1 when $x \in [0, \delta]$ for some $\delta < 1$. Let $Q(x) = x^{-1} - 1$ and for $0 \leq x \leq 1$ define

$$(2.3) \quad \eta(x) = \int_0^x \sqrt{Q(y)} dy.$$

The properties of this function are summarized in the following lemma:

Lemma 2.1. *The map η is a smooth transformation from $[0, 1]$ to $[0, \frac{\pi}{2}]$ and with respect to the change of variables*

$$\omega(\eta) = p^{\frac{1}{2}} f, \quad p = \frac{\eta'}{\eta}$$

(2.2) transforms to

$$(2.4) \quad \sigma^2 \left(\ddot{\omega}(\eta) + \frac{\dot{\omega}(\eta)}{\eta} \right) - \omega(\eta) = \sigma^2 V_-(\eta) \omega(\eta)$$

where

$$V_-(\eta) = \eta^{-1} p^{-\frac{1}{2}} \frac{dp^{\frac{1}{2}}}{d\eta} + p^{-\frac{1}{2}} \frac{d^2 p^{\frac{1}{2}}}{d\eta^2}$$

Here, $\dot{\cdot}$ represents the derivative with respect to η and $'$ the derivative with respect to x .

Furthermore, we may write

$$V_-(\eta) = \frac{1}{\eta^2} + \tilde{V}_-(\eta)$$

for \tilde{V}_- smooth on any interval of the form $[0, \delta]$, $\delta < \frac{\pi}{2}$.

Proof. The smoothness of η is clear. Furthermore, one computes that

$$\dot{\omega} = \eta^{-1} p^{-\frac{1}{2}} f' + \frac{dp^{\frac{1}{2}}}{d\eta} f$$

$$\begin{aligned}\ddot{\omega} &= -\eta^{-2}p^{-\frac{1}{2}}f' - \eta^{-1}p^{-1}\frac{dp^{\frac{1}{2}}}{d\eta}f' + \eta^{-2}p^{-\frac{3}{2}}f'' + \eta^{-1}p^{-1}\frac{dp^{\frac{1}{2}}}{d\eta}f' + \frac{d^2p^{\frac{1}{2}}}{d\eta^2}f \\ &= \eta^{-2}p^{-\frac{3}{2}}f'' - \eta^{-1}\dot{\omega} + \left(\eta^{-1}p^{-\frac{1}{2}}\frac{dp^{\frac{1}{2}}}{d\eta} + p^{-\frac{1}{2}}\frac{d^2p^{\frac{1}{2}}}{d\eta^2} \right) \omega\end{aligned}$$

and thus, using that $\sigma^2 f'' = Qf$ and that $(\eta')^2 = Q$, the above expression for $\ddot{\omega}$ may be rewritten as (2.4).

Furthermore, by the chain rule, one can calculate

$$\begin{aligned}V_-(\eta) &= \frac{1}{4\eta^2} - \frac{3(\eta'')^2}{4(\eta')^4} + \frac{1}{2}\frac{\eta'''}{(\eta')^3} \\ &= \frac{1}{\eta^2} + \left[-\frac{3(\eta'')^2}{4(\eta')^4} + \frac{1}{2}\frac{\eta'''}{(\eta')^3} - \frac{3}{4\eta^2} \right] = \frac{1}{\eta^2} + \tilde{V}_-(\eta).\end{aligned}$$

Moreover, since $\eta(x) = 2x^{\frac{1}{2}} + O_\infty(x^{3/2})$ for $x < 1$, one has $|\partial_\eta^j \tilde{V}_-(\eta)| \lesssim 1$ for $j = 0, 1, \dots$ for $\eta < \pi/2$. \square

With Lemma 2.1 in hand, we look for the solution to (2.4) that is relevant to $e(\sigma, r)$. Recall that the equation (2.1) has a basis of solutions given by $M_{\frac{i}{2\sigma}, \frac{1}{2}}(2ix/\sigma)$, which is a multiple of $e(\sigma, x/\sigma^2)$, and $W_{\frac{i}{2\sigma}, \frac{1}{2}}(2ix/\sigma)$. For any fixed σ , the first one vanishes to first order at $x = 0$ whereas the latter is non-vanishing there [8, (13.14.17)]. Transforming these solutions under the change of variables defined in the above Lemma, the relation $p^{\frac{1}{2}} \sim x^{-\frac{1}{2}}$ shows that (2.4) must have two linearly independent solutions ϕ_- and ϕ_+ satisfying the asymptotics

$$(2.5) \quad \phi_-(\sigma, \eta) \sim \eta \quad \text{and} \quad \phi_+(\sigma, \eta) \sim \eta^{-1}.$$

as $\eta \rightarrow 0$. Therefore, ϕ_- is characterized, up to scaling, as the unique solution to (2.4) that is vanishing (or even finite) at $\eta = 0$. In the following proposition we identify ϕ_- in terms of I_1 and connect it to $e(\sigma, x/\sigma^2)$ using

$$(2.6) \quad \begin{aligned}e(\sigma, x/\sigma^2) &= -i\sigma^{-\frac{1}{2}}[e^{\frac{\pi}{\sigma}} - 1]^{-\frac{1}{2}}M_{\frac{i}{2\sigma}, \frac{1}{2}}(2ix/\sigma) \\ &= 2\sigma^{-\frac{3}{2}}[e^{\frac{\pi}{\sigma}} - 1]^{-\frac{1}{2}}x(1 + O(x/\sigma)) \quad \text{as } x \rightarrow 0.\end{aligned}$$

which follows from (1.10)

Proposition 2.2. *For any $\delta \in (0, \frac{\pi}{2})$, there exists $c > 0$ such that for all $\sigma \in [0, c)$, on $\eta \in [0, \delta]$, $e(\sigma, x/\sigma^2)$ has the form*

$$(2.7) \quad e(\sigma, x/\sigma^2) = 2\sqrt{2}\sigma^{-\frac{1}{2}}[e^{\frac{\pi}{\sigma}} - 1]^{-\frac{1}{2}}\left(\frac{\eta}{\eta'}\right)^{\frac{1}{2}}I_1(\eta/\sigma)(1 + a_-(\sigma, \eta)),$$

where $\rho = \sigma^{-1}\eta$, I_1 is the modified Bessel function of the first kind, and a_- is a smooth function satisfying the bounds

$$|a_-(\sigma, \eta)| \lesssim \sigma, \quad |\dot{a}_-(\sigma, \eta)| \lesssim \sigma, \quad |\ddot{a}_-(\sigma, \eta)| \lesssim 1,$$

$$|\partial_\sigma\{a_-(\sigma, \eta)\}| \lesssim 1, \quad |\partial_\sigma\{\dot{a}_-(\sigma, \eta)\}| \lesssim \sigma^{-2}, \quad |\partial_\sigma^2\{a_-(\sigma, \eta)\}| \lesssim \sigma^{-2}$$

uniformly on $[0, \delta]$.

Remark 2.3. We remark that one can see from the calculations in Lemma 2.1 that $\tilde{V}_- = O((\eta - \pi/2)^{-3})$ as $\eta \rightarrow \pi/2$. This is the main reason why we cannot extend the domain of η past the turning point $\eta = \frac{\pi}{2}$.

Proof. In the variable $\rho = \sigma^{-1}\eta$, the equation (2.4) becomes the perturbed Bessel equation

$$\partial_\rho^2\{\omega(\rho)\} + \rho^{-1}\partial_\rho\{\omega(\rho)\} - (\rho^{-2} + 1)\omega(\rho) = \sigma^2\tilde{V}_-(\sigma\rho)\omega(\rho).$$

A fundamental system for the homogeneous equation

$$\partial_\rho^2\{\omega(\rho)\} + \rho^{-1}\partial_\rho\{\omega(\rho)\} - (\rho^{-2} + 1)\omega(\rho) = 0$$

is given by modified Bessel functions of first order $I_1(\rho), K_1(\rho)$ so that by variation of parameters, the function

$$\phi_-(\sigma, \rho) = I_1(\rho) + \sigma^2 \int_0^\rho \frac{[-I_1(\rho)K_1(u) + K_1(\rho)I_1(u)]\tilde{V}_-(\sigma u)\phi_-(\sigma u)}{W(K_1(u), I_1(u))} du$$

solves (2.4) (provided that the integral on the right converges) and vanishes at $\rho = 0$. Evaluating the Wronskian via [8, (10.28.2)], plugging in the ansatz $\phi_-(\sigma, \rho) = I_1(\rho)(1 + \sigma a_-(\sigma, \rho))$, and noting that $I_1(u)$ has no real zeros we obtain the equation for $a_-(\sigma, \eta)$

$$(2.8) \quad \begin{aligned} a_-(\sigma, \eta) &= \sigma^2 \int_0^\rho u[K_1(u)I_1(u) - I_1^{-1}(\rho)K_1(\rho)I_1^2(u)]\tilde{V}_-(\sigma u)(1 + a_-(\sigma, \sigma u)) du \\ &=: \sigma^2 \int_0^\rho M(\sigma, u)(1 + a_-(\sigma, \sigma u)) du. \end{aligned}$$

We will first prove that $a_-(\sigma, \eta)$ is well-defined and bounded by analyzing the leading term

$$(2.9) \quad a_{-,0}(\sigma, \eta) := \sigma^2 \int_0^\rho u[K_1(u)I_1(u) - I_1^{-1}(\rho)K_1(\rho)I_1^2(u)]\tilde{V}_-(\sigma u) du.$$

For this, we record the following bounds on I_1 and K_1 :

$$(2.10) \quad \begin{aligned} |\partial_u^j\{K_1(u)\}| &\lesssim u^{-1-j}\langle u \rangle^{j+\frac{1}{2}}e^{-u}, \quad j = 0, 1, 2, \dots \\ |\partial_u^j\{I_1(u)\}| &\sim u^{1-j}\langle u \rangle^{j-\frac{3}{2}}e^u, \quad j = 0, 1, \\ |\partial_u^2\{I_1(u)\}| &\lesssim u\langle u \rangle^{-\frac{3}{2}}e^u, \end{aligned}$$

which may easily be deduced from [8, (10.30.1-2) and (10.40.1-2)]. In particular, they imply that

$$|I_1(u)K_1(u)| \lesssim \langle u \rangle^{-1}$$

$$\begin{aligned} |I_1^{-1}(u)K_1(u)| &\lesssim \rho^{-2}\langle \rho \rangle^2 e^{-2\rho} \\ |I_1^2(u)| &\lesssim u^2 \langle u \rangle^{-3} e^{2u}. \end{aligned}$$

Therefore, since by Lemma 2.1 $\tilde{V}_-(\sigma u)$ is bounded for $u \leq \rho$, which is in turn less than $\sigma^{-1}\delta$, we may write

$$\begin{aligned} (2.11) \quad |a_{-,0}(\sigma, \eta)| &\lesssim \sigma^2 \int_0^\rho u \langle u \rangle^{-1} du + \sigma^2 \rho^{-2} \langle \rho \rangle^2 e^{-2\rho} \int_0^\rho u^3 \langle u \rangle^{-3} e^{2u} du \\ &\lesssim \sigma^2 \langle \rho \rangle + \sigma^2 \lesssim \sigma. \end{aligned}$$

Moreover, as $x \rightarrow 0$, we have $|a_{-,0}(\sigma, \eta)| \lesssim \sigma^2 \rho^2 \lesssim x$. These bounds may easily be extended to $a_-(\sigma, \eta)$ itself by a contraction argument. In particular, we simply think of (2.8) as the fixed point equation

$$a_- = T(1) + T(a_-)$$

for the linear operator T given by $Ta = \sigma^2 \int_0^\rho M(\rho, u)a(u)du$. For σ small enough, our computations show that T is a contraction on $L_{\eta}^\infty[0, \delta]$ and moreover that $T(1)$ lies in this space. This implies that the L^∞ norm of the fixed point, given by $a_- = \sum_{n=0}^\infty T^{n+1}(1)$, is bounded by the norm of the first term, which is $O(\sigma)$.

Having established the existence and boundedness of $a_-(\sigma, \eta)$, we now turn to the bounds on its derivatives. We first treat the η -derivative. We have that

$$(2.12) \quad \dot{a}_-(\sigma, \eta) = -\sigma \partial_\rho \{I_1^{-1}(\rho)K_1(\rho)\} \int_0^\rho u I_1^2(u) \tilde{V}_-(\sigma u) (1 + a_-(\sigma, \sigma u)) du.$$

To estimate this integral, we use that by (2.10)

$$|\partial_\rho \{I_1^{-1}(\rho)K_1(\rho)\}| \lesssim \rho^{-3} \langle \rho \rangle^3 e^{-2\rho}$$

so that

$$|\dot{a}_-(\sigma, \eta)| \lesssim \sigma \rho^{-3} \langle \rho \rangle^3 e^{-2\rho} \int_0^\rho u^3 \langle u \rangle^{-3} e^{2u} du \lesssim \sigma.$$

Differentiating again, we find that

$$\begin{aligned} \ddot{a}_-(\sigma, \eta) &= -\partial_\rho \{I_1^{-1}(\rho)K_1(\rho)\} \rho I_1^2(\rho) \tilde{V}_-(\eta) (1 + a_-(\sigma, \eta)) \\ &\quad - \partial_\rho^2 \{I_1^{-1}(\rho)K_1(\rho)\} \int_0^\rho u I_1^2(u) \tilde{V}_-(\sigma u) (1 + a_-(\sigma, \sigma u)) du. \end{aligned}$$

By (2.10), the first term is uniformly bounded. For the second, we use

$$|\partial_\rho^2 \{I_1^{-1}(\rho)K_1(\rho)\}| \lesssim \rho^{-4} \langle \rho \rangle^4 e^{-2\rho}$$

to argue similarly that the second term is uniformly bounded as well.

Finally, we estimate the σ -derivatives of $a(\sigma, \eta)$. For the first derivative, we have

$$\begin{aligned}
\partial_\sigma \{a_-(\sigma, \eta)\} &= \sigma^{-1}(2a_-(\sigma, \eta) - \eta \dot{a}(\sigma, \eta)) \\
&+ \sigma^2 \int_0^\rho u^2 [K_1(u) - I_1^{-1}(\rho)K_1(\rho)I_1(u)] \tilde{V}'_-(\sigma u) I_1(u) (1 + a_-(\sigma, \sigma u)) du \\
&+ \sigma^2 \int_0^\rho u^2 [K_1(u) - I_1^{-1}(\rho)K_1(\rho)I_1(u)] \tilde{V}_-(\sigma u) I_1(u) \dot{a}_-(\sigma, \sigma u) du \\
&+ \sigma^2 \int_0^\rho u [K_1(u) - I_1^{-1}(\rho)K_1(\rho)I_1(u)] \tilde{V}_-(\sigma u) I_1(u) \partial_\sigma \{a_-(\sigma, x)\}|_{x=\sigma u} du. \\
&=: A_1(\sigma, \eta) + A_2(\sigma, \eta) + A_3(\sigma, \eta) + A_4(\sigma, \eta)
\end{aligned}$$

By the bounds on $a(\sigma, \eta)$ and $\dot{a}(\sigma, \eta)$, it is clear $|A_1(\sigma, \eta)| \lesssim 1$. Since \tilde{V}_- is uniformly smooth by Lemma 2.1, it is easy to argue as for the bound $|a_{-,0}| \lesssim \sigma$ that $|A_2(\sigma, \eta)| \lesssim 1$, the only difference being an extra power of u in the integral. Similarly, $|A_3(\sigma, \eta)| \lesssim \sigma$ due to the previously derived bound $|\dot{a}(\sigma, \eta)| \lesssim \sigma$.

We have shown then that $\partial_\sigma \{a(\sigma, \eta)\}$ satisfies a fixed point equation of the form

$$\partial_\sigma \{a_-(\sigma, \eta)\} = O(1) + \sigma^2 \int_0^\rho u [K_1(u) - I_1^{-1}(\rho)K_1(\rho)I_1(u)] \tilde{V}_-(\sigma u) I_1(u) \partial_\sigma \{a_-(\sigma, \sigma u)\} du$$

and since we have already shown via (2.11) that the last term is bounded in terms of $\sigma \sup_{\eta \in [0, \delta]} |\partial_\sigma \{a_-(\sigma, \sigma u)\}|$, we may iterate this equation to find that, for σ sufficiently small, $\partial_\sigma \{a(\sigma, \eta)\}$ is uniformly bounded independent of σ on the domain under consideration.

For the mixed σ and η derivative, we first compute that

$$\begin{aligned}
\partial_\sigma \{\dot{a}(\sigma, \eta)\} &= \sigma^{-1} \dot{a}(\sigma, \eta) - \sigma^{-1} \eta \ddot{a}_-(\sigma, \eta) \\
&- \sigma \partial_\sigma \{I_1^{-1}(\rho)K_1(\rho)\} \int_0^\rho u^2 I_1^2(u) \tilde{V}'_-(\sigma u) (1 + a_-(\sigma, \sigma u)) du \\
&- \sigma \partial_\sigma \{I_1^{-1}(\rho)K_1(\rho)\} \int_0^\rho u^2 I_1^2(u) \tilde{V}_-(\sigma u) \dot{a}_-(\sigma, \sigma u) du \\
&- \sigma \partial_\sigma \{I_1^{-1}(\rho)K_1(\rho)\} \int_0^\rho u I_1^2(u) \tilde{V}_-(\sigma u) \partial_\sigma \{a_-(\sigma, x)\}|_{x=\sigma u} du.
\end{aligned}$$

The first two terms are bounded by σ^{-1} and the third and fourth are bounded by a constant times

$$\sigma \rho^{-3} \langle \rho \rangle^3 e^{-2\rho} \int_0^\rho u^4 \langle u \rangle^{-3} e^{2u} du \lesssim \sigma \langle \rho \rangle^2 \lesssim \sigma^{-1}.$$

Similarly, the last term is $O(\sigma^{-1})$ from the fact that $\partial_\sigma\{a(\sigma, \eta)\}$ is bounded.

The second σ -derivative is now estimated by differentiating each of A_i for $i = 1, 2, 3, 4$ in turn. It is easy to see from the bounds we have already developed that

$$|\partial_\sigma\{A_1(\sigma, \eta)\}| \lesssim \sigma^{-2}$$

the dominant term being $\sigma^{-1}\eta\partial_\sigma\{\dot{a}(\sigma, \eta)\}$. Furthermore,

$$\begin{aligned} |\partial_\sigma\{A_2(\sigma, \eta)\}| &\lesssim \sigma^{-1}|A_2(\sigma, \eta)| + \left| \eta\partial_\sigma\{I_1^{-1}(\rho)K_1(\rho)\} \int_0^\rho u^2 I_1^2(u) du \right| \\ &\quad + \sigma^2 \left| \int_0^\rho u^3 [-K_1(u)I_1(u) + I_1^{-1}(\rho)K_1(\rho)I_1^2(u)] du \right| \end{aligned}$$

which is in total bounded in terms of σ^{-1} . By the same token,

$$\begin{aligned} |\partial_\sigma\{A_3(\sigma, \eta)\}| &\lesssim \sigma^{-1}|A_3(\sigma, \eta)| + \sigma \left| \eta\partial_\sigma\{I_1^{-1}(\rho)K_1(\rho)\} \int_0^\rho u^2 I_1^2(u) du \right| \\ &\quad + \sigma \left| \int_0^\rho u^2 \langle u \rangle [K_1(u)I_1(u) - I_1^{-1}(\rho)K_1(\rho)I_1^2(u)] du \right| \end{aligned}$$

where we have used that $|\partial_\sigma\{\dot{a}(\sigma, \eta)\}| \lesssim \sigma^{-1}$. As before, we may argue that all three terms are bounded by a constant times σ^{-1} . One can also show that

$$A_4(\sigma, \eta) = O(\sigma^{-1}) + \sigma^2 \int_0^\rho u [K_1(u) - I_1^{-1}(\rho)K_1(\rho)I_1(u)] \tilde{V}_-(\sigma u) I_1(u) \partial_\sigma^2\{a_-(\sigma, \sigma u)\} du$$

so that

$$\partial_\sigma^2\{a_-(\sigma, \sigma u)\} = O(\sigma^{-1}) + \sigma^2 \int_0^\rho u [K_1(u) - I_1^{-1}(\rho)K_1(\rho)I_1(u)] \tilde{V}_-(\sigma u) I_1(u) \partial_\sigma^2\{a_-(\sigma, \sigma u)\} du.$$

Since the last term is bounded in terms of $\sigma \sup_{\eta \in [0, \delta]} |\partial_\sigma\{a_-(\sigma, \sigma u)\}|$, as before we can iterate for small enough σ to see that in fact $\partial_\sigma^2\{a_-(\sigma, \eta)\} = O(\sigma^{-1})$, as claimed.

Finally, we match the solution $\phi_-(\sigma, \eta)$ to $e(\sigma, x/\sigma^2)$. For any fixed σ , as $x \rightarrow 0$ we have

$$\left(\frac{\eta(x)}{\eta'(x)} \right)^{\frac{1}{2}} \phi_-(\rho) = \frac{\sqrt{2}x}{\sigma} (1 + O(x/\sigma)).$$

as a consequence of [8, (10.30.1)]. Comparing this expansion to (2.6) we obtain (2.7). \square

Before we focus on the other regimes of x , we give the following bounds which will be useful for the oscillatory estimates.

Lemma 2.4. *For any $\delta \in (0, 1)$ there exists $c > 0$ and $\varepsilon > 0$ so that*

$$(2.13) \quad |\partial_\sigma^j \{e(\sigma, r)\}| \lesssim e^{-\frac{\varepsilon}{\sigma} r}, \quad j = 0, 1, 2$$

uniformly for $\sigma \in [0, \min\{c, \delta r^{-\frac{1}{2}}\}]$ and $r > 0$.

Proof. We will use throughout the proof that we are considering σ so that $x = \sigma^2 r \leq \delta < 1$. First, this allows us to apply Proposition (2.2) on the interval $[0, \eta(\delta)]$ to obtain the representation (2.7). By series expansion,

$$\eta(\sigma^2 r) = 2(\sigma^2 r)^{\frac{1}{2}} - \frac{1}{3}(\sigma^2 r)^{\frac{3}{2}} + O_2((\sigma^2 r)^{\frac{5}{2}})$$

which gives

$$\left(\frac{\eta}{\eta'}\right)^{\frac{1}{2}}(\sigma^2 r) = \sqrt{2}\sigma r^{\frac{1}{2}}(1 + O_2(\sigma^2 r)).$$

Therefore, using (2.10) we obtain

$$|I_1\left(\frac{\eta(\sigma^2 r)}{\sigma}\right)| \lesssim \min\left\{1, \left|\frac{\eta(\sigma^2 r)}{\sigma}\right|\right\} e^{\frac{\eta(\sigma^2 r)}{\sigma}} \lesssim e^{\frac{\pi}{2\sigma} - \frac{\varepsilon}{\sigma}} \min\{1, r^{\frac{1}{2}}\}.$$

for $\varepsilon < \frac{\pi}{2} - \eta(\delta)$. Hence, by (2.7)

$$|e(\sigma, \sigma r)| \lesssim \sigma^{-\frac{1}{2}} [e^{\frac{\pi}{\sigma}} - 1]^{-\frac{1}{2}} \sigma r^{\frac{1}{2}} |I_1\left(\frac{\eta(\sigma^2 r)}{\sigma}\right)| (1 + a_-(\sigma, \eta)) \lesssim e^{-\frac{\varepsilon}{\sigma} r^{\frac{1}{2}}} \min\{1, r^{\frac{1}{2}}\} \lesssim e^{-\frac{\varepsilon}{\sigma} r}.$$

This establishes (2.13) for $j = 0$.

Next, we estimate the σ -derivative. For the remainder of the proof we suppress the dependence of η on $\sigma^2 r$. By the chain rule one has

$$\partial_\sigma \{I_1(\eta/\sigma)\} = I'(\eta/\sigma) [-\sigma^{-2}\eta + \sigma^{-1} \frac{d\eta}{d\sigma}].$$

and we have $|\frac{d\eta}{d\sigma}| \lesssim r^{\frac{1}{2}}$ for $\sigma \lesssim r^{-\frac{1}{2}}$. Therefore, by (2.10)

$$|\partial_\sigma \{I_1(\eta/\sigma)\}| \lesssim e^{\frac{\pi}{2\sigma} - \frac{\varepsilon}{\sigma}} \sigma^{-1} r^{\frac{1}{2}}$$

and

$$(2.14) \quad |\partial_\sigma \left\{ \left(\frac{\eta}{\eta'}\right)^{\frac{1}{2}} I_1(\eta/\sigma) \right\}| \lesssim e^{\frac{\pi}{2\sigma} - \frac{\varepsilon}{\sigma}} r.$$

Now, we note that,

$$(2.15) \quad \partial_\sigma \{a_-(\sigma, \eta)\} = \partial_\sigma \{a_-(\sigma, \eta(x))\}|_{x=\sigma^2 r} + \dot{a}_-(\sigma, \eta) \frac{d\eta}{d\sigma}.$$

Therefore,

$$(2.16) \quad |\partial_\sigma \{a_-(\sigma, \eta(\sigma^2 r))\}| \lesssim 1 + \sigma r^{\frac{1}{2}} \lesssim 1.$$

Using (2.14) and (2.16) in (2.7) and applying the product rule, we obtain (2.13) for $j = 1$.

For the second derivative in σ , we start by computing

$$\partial_\sigma^2 \{I_1(\eta/\sigma)\} = I_1''(\eta/\sigma) [-\sigma^{-2}\eta + \sigma^{-1} \frac{d\eta}{d\sigma}]^2 + I_1'(\eta/\sigma) [2\sigma^{-3}\eta - 2\sigma^{-2} \frac{d\eta}{d\sigma} + \sigma^{-1} \frac{d^2\eta}{d\sigma^2}].$$

Using (2.10) for $j = 2$, we have

$$|I_1''(\eta/\sigma) [-\sigma^{-2}\eta + \sigma^{-1} \frac{d\eta}{d\sigma}]^2| \lesssim e^{\frac{\pi}{2\sigma} - \frac{\varepsilon}{\sigma}} (\sigma^{-1} r^{\frac{1}{2}})^2.$$

Moreover, $|\frac{d^2\eta}{d\sigma^2}| \lesssim \sigma^{-1} r^{\frac{1}{2}}$, and $|I_1'(\eta/\sigma)| \lesssim e^{\frac{\pi}{2\sigma} - \frac{\varepsilon}{\sigma}}$. Therefore, we have

$$|\partial_\sigma^2 \{I_1(\eta/\sigma)\}| \lesssim e^{\frac{\pi}{2\sigma} - \frac{\varepsilon}{\sigma}} \sigma^{-2} r$$

and by the chain rule

$$(2.17) \quad \left| \partial_\sigma^2 \left\{ \left(\frac{\eta}{\eta'} \right)^{\frac{1}{2}} I_1 \left(\frac{\eta}{\sigma} \right) \right\} \right| \lesssim e^{\frac{\pi}{2\sigma} - \frac{\varepsilon}{\sigma}} [\sigma^{-2} r + \sigma^{-1} r] \lesssim e^{\frac{\pi}{2\sigma} - \frac{\varepsilon}{\sigma}} \sigma^{-2} r.$$

In the last line we used the fact that $\sigma^2 r < c$ and $\sigma < c$. Moreover,

$$(2.18) \quad \begin{aligned} |\partial_\sigma^2 \{a_-(\sigma, \eta(\sigma^2 r))\}| &\lesssim |\partial_\sigma^2 \{a_-(\sigma, \eta(x))\}|_{x=\sigma^2 r} + |\partial_\sigma \{\dot{a}_-(\sigma, \eta)\} \frac{d\eta}{d\sigma}| \\ &\quad + |\ddot{a}_-(\sigma, \eta) \left(\frac{d\eta}{d\sigma} \right)^2| + |\dot{a}_-(\sigma, \eta) \frac{d^2\eta}{d\sigma^2}| \end{aligned}$$

and hence $|\partial_\sigma^2 \{a_-(\sigma, \eta(\sigma^2 r))\}| \lesssim \sigma^{-2} + \sigma^{-2} r^{\frac{1}{2}} + r \lesssim \sigma^{-3}$. Finally, the chain rule together with (2.17) and (2.18) gives (2.13) for $j = 2$. □

2.2. Airy function approximation: $x \sim 1$. Let $Q(u) = u^{-1} - 1$ and define the *Liouville-Green transform*

$$\zeta(x) = \text{sign}(x - 1) \left| \frac{3}{2} \int_1^x \sqrt{|Q(u)|} du \right|^{\frac{2}{3}}.$$

Its properties are summarized in the following lemmas:

Lemma 2.5. *The map $x \mapsto \zeta(x)$ defines a smooth change of variables from $(0, \infty) \rightarrow (-\frac{3\pi}{4})^{\frac{2}{3}}, \infty)$. Furthermore, ζ has the explicit form given by*

$$(2.19) \quad \begin{aligned} \frac{2}{3} \zeta^{\frac{3}{2}}(x) &= \sqrt{x(x-1)} - \log(\sqrt{x} + \sqrt{x-1}), \quad x \geq 1, \\ -\frac{2}{3} (-\zeta(x))^{\frac{3}{2}} &= \sqrt{x(1-x)} - \frac{1}{2} \arccos(2x-1), \quad x \leq 1. \end{aligned}$$

The function $q = -\frac{Q}{\zeta}$ is non-negative and satisfies $\sqrt{q} = \frac{d\zeta}{dx}$. Under the transformation $w(\zeta) = q^{\frac{1}{4}} f$, the equation (2.2) becomes

$$(2.20) \quad -\sigma^2 \ddot{w}(\zeta) = (\zeta + \sigma^2 V) w(\zeta)$$

where

$$V = -q^{-\frac{1}{4}} \frac{d^2 q^{\frac{1}{4}}}{d\zeta^2}.$$

Here and for the rest of the paper we use $\dot{\cdot} = \frac{d}{d\zeta}$.

Proof. The smoothness of ζ is clear away from $x = 1$ and at this point it is a simple consequence of the fact that Q vanishes only to first order. Indeed, we may expand $\sqrt{|Q|}$ into a series in powers of $\sqrt{|x-1|}$ and integrate term by term to find that

$$\int_1^x \sqrt{|Q(u)|} du = \frac{2}{3}(x-1)^{\frac{3}{2}}(1 + O(x-1))$$

from which the claim is immediate. We omit the proof of (2.19) and (2.20) as it can be verified by differentiation. □

For reference, we record all of the notation relevant to the Liouville-Green transform:

$$\begin{aligned} Q(u) &= u^{-1} - 1, \quad q = -\frac{Q}{\zeta} \\ \zeta(x) &= \text{sign}(x-1) \left| \frac{3}{2} \int_1^x \sqrt{|Q(u)|} du \right|^{\frac{2}{3}}. \\ w &= q^{\frac{1}{4}} f, \quad V = -q^{-\frac{1}{4}} \frac{d^2 q^{\frac{1}{4}}}{d\zeta^2} \end{aligned}$$

Lemma 2.6. *Let $\zeta^* = -(3\pi/4)^{\frac{2}{3}}$ and $\zeta \in (\zeta^*, 0]$ then we have $|\partial_\zeta^j V| \lesssim 1$ for $j = 0, 1, 2, \dots$.*

Proof. We note that as $|x-1| < 1$, one has $\zeta \sim (x-1)$, and therefore, $q^{\frac{1}{4}} = \sum_{k=0}^{\infty} c_k \zeta^k$ for some $c_k \in \mathbb{R}$. This shows that $|\partial_\zeta^j V| \lesssim 1$ in the range of $|\zeta| < 1$. On the other hand, as $|x| < 1$, one has $(\zeta - \zeta^*)^{\frac{3}{2}} \sim x$, therefore, $q^{\frac{1}{4}} \sim (\zeta - \zeta^*)^{-\frac{3}{8}}$ and $V(\zeta) \in O_\infty((\zeta - \zeta^*)^{-2})$. This shows that $|\partial_\zeta^j V| \lesssim 1$ as long as $|\zeta - \zeta^*| > \delta > 0$. □

We may now construct a basis of solutions to (2.20) in terms of the Airy functions Ai and Bi whose properties may be found in [32].

Proposition 2.7. *Let $\delta > 0$. Then there exists $c > 0$ such that for all $\sigma \in [0, c]$, a fundamental system of solutions to (2.20) in the range $\zeta^* + \delta < \zeta \leq 0$ is given by*

$$(2.21) \quad \begin{aligned} \phi_1(\sigma, \zeta) &= \text{Ai}(\tau)(1 + \sigma a_1(\sigma, \zeta)) \\ \phi_2(\sigma, \zeta) &= \text{Bi}(\tau)(1 + \sigma a_2(\sigma, \zeta)) \end{aligned}$$

where $\tau := -\sigma^{-\frac{2}{3}}\zeta$ and a_1 and a_2 are smooth functions satisfying the bounds

$$|a_j(\sigma, \zeta)| \lesssim 1, \quad |\dot{a}_j(\sigma, \zeta)| \lesssim \sigma^{-\frac{1}{3}}, \quad |\ddot{a}_j(\sigma, \zeta)| \lesssim \sigma^{-\frac{4}{3}},$$

$$|\partial_\sigma\{a_j(\sigma, \zeta)\}| \lesssim \sigma^{-\frac{4}{3}}, \quad |\partial_\sigma\{\dot{a}_j(\sigma, \zeta)\}| \lesssim \sigma^{-7/3}, \quad |\partial_\sigma^2\{a_j(\sigma, \zeta)\}| \lesssim \sigma^{-\frac{10}{3}}$$

for $j = 1, 2$ uniformly on $[\zeta_* + \delta, 0]$.

Remark 2.8. *The range of ζ corresponds to $x \in [\delta', 1]$ for some $\delta' > 0$ independent of σ . The restriction is designed to avoid the singularity of V at $\zeta = \zeta_*$, see the proof of Lemma 2.6. Note also that this approximation is only possible because $\tau > 0$ for $\zeta < 0$ and thus the Airy functions do not have zeroes in this regime.*

Proof. Write $\phi_{1,0}(\sigma, \zeta) = \text{Ai}(\tau)$ and $\phi_{2,0}(\sigma, \zeta) = \text{Bi}(\tau)$. The variable τ is chosen so that

$$-\sigma^2 \ddot{\phi}_{j,0} - \zeta \phi_{j,0} = 0$$

for each of $j = 1, 2$ where $\dot{\cdot} = \frac{\partial}{\partial \zeta}$. Therefore,

$$-\sigma^2 \ddot{\phi}_j - \zeta \phi_j = (\sigma^3 \phi_{j,0}^2 \dot{a}_j) \dot{\cdot} / \phi_{j,0}$$

and plugging the representations (2.21) into (2.20) yields the equation for a_j

$$(2.22) \quad (\phi_{j,0}^2 \dot{a}_j) \dot{\cdot} = -\sigma^{-1} V \phi_{j,0}^2 (1 + \sigma a_j).$$

The solution to this equation for $j = 2$ with $a_2(\sigma, 0) = 0$ and $\dot{a}_2(\sigma, 0) = 0$ is given by

$$(2.23) \quad a_2(\sigma, \zeta) = -\sigma^{\frac{1}{3}} \int_0^{-\sigma^{-\frac{2}{3}}\zeta} \text{Bi}^2(u) \left[\int_u^{-\sigma^{-\frac{2}{3}}\zeta} \text{Bi}^{-2}(v) dv \right] V(-\sigma^{\frac{2}{3}}u) (1 + \sigma a_2(\sigma, -\sigma^{\frac{2}{3}}u)) du.$$

We now recall the following expansions of the Airy functions found in [32]:

$$(2.24) \quad \text{Bi}(x) = \pi^{-\frac{1}{2}} x^{-\frac{1}{4}} e^{\frac{2}{3}x^{\frac{3}{2}}} (1 + O(x^{-\frac{3}{2}})) \text{ as } x \rightarrow \infty$$

$$\text{Bi}(x) \geq \text{Bi}(0) > 0, \quad \text{for } x \geq 0$$

$$(2.25) \quad \text{Ai}(x) = \frac{1}{2} \pi^{-\frac{1}{2}} x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{\frac{3}{2}}} (1 + O(x^{-\frac{3}{2}})) \text{ as } x \rightarrow \infty$$

$$\text{Ai}(x) > 0 \quad \text{for } x \geq 0$$

These asymptotics and the identity, which holds for $0 \leq x_0 < x_1$

$$(2.26) \quad \int_{x_0}^{x_1} \text{Bi}^{-2}(y) dy = \pi^{-1} \left(\frac{\text{Ai}}{\text{Bi}}(x_0) - \frac{\text{Ai}}{\text{Bi}}(x_1) \right)$$

imply that for $x_0 \geq 0$

$$\left| \text{Bi}^2(x_0) \int_{x_0}^{x_1} \text{Bi}^{-2}(y) dy \right| \lesssim \langle x_0 \rangle^{-\frac{1}{2}}$$

so that

$$(2.27) \quad \left| \int_0^{-\sigma^{-\frac{2}{3}}\zeta} \text{Bi}^2(u) \left[\int_u^{-\sigma^{-\frac{2}{3}}\zeta} \text{Bi}^{-2}(v) dv \right] f(u) du \right| \lesssim \langle \sigma^{-\frac{2}{3}}\zeta \rangle^{\frac{1}{2}} \|f\|_\infty.$$

Moreover,

$$(2.28) \quad \left| \text{Bi}^{-2}(x_0) \int_0^{x_0} \text{Bi}^2(y) dy \right| \lesssim \langle x_0 \rangle^{-\frac{1}{2}}$$

which comes from inserting the above asymptotics into the integral and then computing for $x > 1$

$$\begin{aligned} x^{\frac{1}{2}} e^{-\frac{4}{3}x^{\frac{3}{2}}} \int_1^x y^{-\frac{1}{2}} e^{\frac{4}{3}y^{\frac{3}{2}}} dy &\lesssim x^{\frac{1}{2}} e^{-\frac{4}{3}x^{\frac{3}{2}}} \int_1^{x^{\frac{3}{2}}} u^{-\frac{2}{3}} e^{\frac{4}{3}u} du = x^{\frac{1}{2}} e^{-\frac{4}{3}x^{\frac{3}{2}}} \left(\frac{3}{4} u^{-\frac{2}{3}} e^{\frac{4}{3}u} \Big|_1^{x^{\frac{3}{2}}} + \frac{2}{3} \int_1^{x^{\frac{3}{2}}} u^{-\frac{5}{3}} e^{\frac{4}{3}u} du \right) \\ &\lesssim x^{\frac{1}{2}} e^{-\frac{4}{3}x^{\frac{3}{2}}} \left(x^{-1} e^{\frac{4}{3}x^{\frac{3}{2}}} \right) = x^{-\frac{1}{2}} \end{aligned}$$

To estimate (2.23), we let

$$a_{2,0}(\zeta) := -\sigma^{\frac{1}{3}} \int_0^{-\sigma^{-\frac{2}{3}}\zeta} \text{Bi}^2(u) \left[\int_u^{-\sigma^{-\frac{2}{3}}\zeta} \text{Bi}^{-2}(v) dv \right] V(-\sigma^{\frac{2}{3}}u) du$$

be the leading term, where we have suppressed the σ -dependence of a_2 for now. By Lemma 2.6 $V(-\sigma^{\frac{2}{3}}u)$ is bounded on the domain of integration when $\zeta \in [\zeta_* + \delta, 0]$ so using (2.27) we have that

$$(2.29) \quad |a_{2,0}(\zeta)| \lesssim \sigma^{\frac{1}{3}} \langle \sigma^{-\frac{2}{3}}\zeta \rangle^{\frac{1}{2}} \lesssim 1.$$

Now, a contraction argument will show that $|a_2(\zeta)| \lesssim 1$, as claimed.

We next consider the ζ derivative of $a_2(\zeta)$. One has that

$$(2.30) \quad \dot{a}_2(\zeta) = \sigma^{-\frac{1}{3}} \text{Bi}^{-2}(-\sigma^{-\frac{2}{3}}\zeta) \int_0^{-\sigma^{-\frac{2}{3}}\zeta} \text{Bi}^2(u) V(-\sigma^{\frac{2}{3}}u) (1 + \sigma a_2(-\sigma^{\frac{2}{3}}u)) du$$

so that, since V is bounded and we have shown that a_2 itself is bounded, we see that by (2.28)

$$|\dot{a}_2(\zeta)| \lesssim \sigma^{-\frac{1}{3}} \langle \sigma^{-\frac{2}{3}}\zeta \rangle^{-\frac{1}{2}},$$

which is less than $\sigma^{-\frac{1}{3}}$, as claimed.

For the second ζ -derivative, we use (2.22) to write

$$\ddot{a}_2(\zeta) = -\sigma^{-1} V(\zeta) (1 + \sigma a_2(\zeta)) - 2\sigma^{-1} [\dot{\phi}_{2,0} \phi_{2,0}^{-1}](\zeta) \dot{a}_2(\zeta).$$

The first term is clearly bounded by σ^{-1} while the second is bounded in terms of

$$\sigma^{-1} |\text{Bi}'(-\sigma^{-\frac{2}{3}}\zeta) \text{Bi}^{-1}(-\sigma^{-\frac{2}{3}}\zeta)| \lesssim \sigma^{-\frac{4}{3}}.$$

By [8, (9.7.8)], $|\text{Bi}'(-\sigma^{-\frac{2}{3}}\zeta)| \lesssim \langle \sigma^{-\frac{2}{3}}\zeta \rangle^{\frac{1}{4}} e^{\frac{2}{3}\sigma^{-1}|\zeta|^{\frac{3}{2}}}$, which shows that $|\ddot{a}_2(\zeta)| \lesssim \sigma^{-\frac{4}{3}}$.

Now, we consider the σ -derivatives of a_2 . Let $F(\sigma, u) = V(-\sigma^{\frac{2}{3}}u)(1 + \sigma a_2(\sigma, -\sigma^{\frac{2}{3}}u))$, then by (2.23) we can compute

$$\begin{aligned} \partial_\sigma \{a_2(\sigma, \zeta)\} &= -\frac{1}{3\sigma} [a_2(\sigma, \zeta) - 2\zeta \dot{a}_2(\sigma, \zeta)] \\ &\quad - \sigma^{\frac{1}{3}} \int_0^{-\sigma^{-\frac{2}{3}}\zeta} \text{Bi}^2(u) \left[\int_u^{-\sigma^{-\frac{2}{3}}\zeta} \text{Bi}^{-2}(v) dv \right] \partial_\sigma \{F(u, \sigma)\} d\sigma =: B_1(\sigma, \zeta) + B_2(\sigma, \zeta) \end{aligned}$$

Using the bounds previously established for $a_2(\sigma, \zeta)$, we can deduce $|B_1(\sigma, \zeta)| \lesssim \sigma^{-\frac{4}{3}}$. Moreover, we may write

$$(2.31) \quad \partial_\sigma \{F(u, \sigma)\} = O(\sigma^{-\frac{1}{3}} \langle u \rangle) + \sigma V(-\sigma^{\frac{3}{2}}u) \partial_\sigma [a_2(\sigma, v)]_{v=-\sigma^{\frac{3}{2}}u}.$$

Therefore, by (2.27) we have

$$B_2(\sigma, \zeta) = \sigma^{\frac{1}{3}} \langle \sigma^{-\frac{2}{3}}\zeta \rangle^{\frac{3}{2}} + -\sigma^{\frac{4}{3}} \int_0^{-\sigma^{-\frac{2}{3}}\zeta} \text{Bi}^2(u) \left[\int_u^{-\sigma^{-\frac{2}{3}}\zeta} \text{Bi}^{-2}(v) dv \right] V(-\sigma^{\frac{3}{2}}u) \partial_\sigma [a_2(\sigma, v)]_{v=-\sigma^{\frac{3}{2}}u} du$$

Letting

$$T(a) := \sigma^{\frac{1}{3}} \int_0^{-\sigma^{-\frac{2}{3}}\zeta} \text{Bi}^2(u) \left[\int_u^{-\sigma^{-\frac{2}{3}}\zeta} \text{Bi}^{-2}(v) dv \right] V(-\sigma^{\frac{3}{2}}u) a(\sigma, u) du$$

we obtain

$$(2.32) \quad \partial_\sigma \{a_2(\sigma, \zeta)\} = \sigma^{-1} + \sigma T \left(\partial_\sigma [a_2(\sigma, v)]_{v=-\sigma^{\frac{3}{2}}u} \right)$$

Now, by contraction argument we obtain that $|\partial_\sigma \{a_2(\sigma, \zeta)\}| \lesssim \sigma^{-\frac{4}{3}}$.

Proceeding onward, we differentiate (2.30) with respect to σ to find that

$$\begin{aligned} \partial_\sigma \{\dot{a}_2(\sigma, \zeta)\} &= -\frac{1}{3}\sigma^{-1} \dot{a}_2(\sigma, \zeta) + \frac{2}{3}\sigma^{-\frac{5}{3}}\zeta \text{Bi}'(-\sigma^{-\frac{2}{3}}\zeta) \text{Bi}^{-1}(-\sigma^{-\frac{2}{3}}\zeta) \dot{a}_2(\sigma, \zeta) \\ &\quad + \frac{2}{3}\sigma^{-2}\zeta V_+(\zeta)(1 + \sigma a_2(\sigma, \zeta)) \\ &\quad + \sigma^{-\frac{1}{3}} \text{Bi}^{-2}(-\sigma^{-\frac{2}{3}}\zeta) \int_0^{-\sigma^{-\frac{2}{3}}\zeta} \text{Bi}^2(u) \partial_\sigma \{F(u, \sigma)\} du. \end{aligned}$$

It is easy to see using previously derived inequalities that each term is at least $O(\sigma^{-2})$ except for the second term. For that, we write

$$|\sigma^{-\frac{5}{3}}\zeta \text{Bi}'(-\sigma^{-\frac{2}{3}}\zeta) \text{Bi}^{-1}(-\sigma^{-\frac{2}{3}}\zeta)\dot{a}_2(\sigma, \zeta)| \lesssim \sigma^{-2}\zeta \left\langle \sigma^{-\frac{2}{3}}\zeta \right\rangle^{\frac{1}{2}} |\dot{a}_2(\sigma, \zeta)|$$

and recall that $|\dot{a}_2(\sigma, \zeta)| \lesssim \sigma^{-\frac{1}{3}} \left\langle \sigma^{-\frac{2}{3}}\zeta \right\rangle^{-\frac{1}{2}}$. It follows then that $|\partial_\sigma \{\dot{a}_2(\sigma, \zeta)\}| \lesssim \sigma^{-2}$.

For the second σ -derivative of a_2 , we differentiate each of $B_j(\sigma, \zeta)$ $j = 1, 2$ separately. First, it is easy to see that

$$|\partial_\sigma \{B_1(\sigma, \zeta)\}| \lesssim \sigma^{-1}|B_1(\sigma, \zeta)| + \sigma^{-1}(|\partial_\sigma \{a_2(\sigma, \zeta)\}| + \sigma^{-1}|\partial_\sigma \{\dot{a}_2(\sigma, \zeta)\}|)$$

which is in total $O(\sigma^{-3})$, the dominant term being the last term. Next, differentiating $B_2(\sigma, \zeta)$ we have

$$(2.33) \quad \begin{aligned} \partial_\sigma \{B_2(\sigma, \zeta)\} &= \frac{1}{3\sigma}B_2(\sigma, \zeta) + \frac{2}{3}\sigma^{-\frac{4}{3}}\zeta \text{Bi}^{-2}(-\sigma^{-\frac{2}{3}}\zeta) \int_0^{-\sigma^{-\frac{2}{3}}\zeta} \text{Bi}^2(u) \partial_\sigma \{F(u, \sigma)\} du \\ &\quad + \sigma^{\frac{1}{3}} \int_0^\tau \text{Bi}^2(u) \left[\int_u^\tau \text{Bi}^{-2}(v) dv \right] \partial_\sigma^2 \{F(u, \sigma)\} du. \end{aligned}$$

Similar calculations to those employed in the estimation of $B_2(\sigma, \zeta)$ demonstrate that the initial two terms on the right-hand side of the equation in (2.33) are $O(\sigma^{-\frac{7}{3}})$. To estimate the last term, we compute

$$\partial_\sigma^2 \{F(u, \sigma)\} = O(\sigma^{-1}\langle u \rangle^2 + \sigma^2) + \sigma V(-\sigma^{\frac{2}{3}}u) \partial_\sigma^2 [a_2(\sigma, v)]_{v=-\sigma^{\frac{2}{3}}u}.$$

Hence, using (2.28), we deduce the following expressions:

$$\partial_\sigma \{B_2(\sigma, \zeta)\} = O(\sigma^{-\frac{7}{3}}) + \sigma T \left(\partial_\sigma^2 [a_2(\sigma, v)]_{v=-\sigma^{\frac{2}{3}}u} \right).$$

This, in turn, leads to:

$$\partial_\sigma^2 [a_2(\sigma, \zeta)] = O(\sigma^{-3}) + \sigma T \left(\partial_\sigma^2 [a_2(\sigma, v)]_{v=-\sigma^{\frac{2}{3}}u} \right).$$

which with a contraction argument, yields $|\partial_\sigma^2 [a_2(\sigma, \zeta)]| \lesssim \sigma^{-3}$.

Having proven all of the stated bounds on a_2 , we now turn to a_1 . We make the reduction ansatz $\phi_1(\zeta) = g(\zeta)\phi_2(\zeta)$ and find that g solves $(\phi_2^2 \dot{g}) = 0$. To simplify the analysis that follows, we extend the functions ϕ_1 and ϕ_2 , defined at the moment on $[\zeta_* + \delta, 0]$, to the interval $(-\infty, 0]$ in such a way that the proven bounds still hold. We then choose the solution g of the form

$$g(\zeta) = \pi \int_\tau^\infty \text{Bi}^{-2}(u) (1 + a_2(-\sigma^{\frac{2}{3}}u))^{-2} du,$$

which yields

$$(2.34) \quad \phi_1(\zeta) = \pi \operatorname{Bi}(\tau)(1 + \sigma a_2(\zeta)) \int_{\tau}^{\infty} \operatorname{Bi}^{-2}(u)(1 + \sigma a_2(-\sigma^{\frac{2}{3}}u))^{-2} du.$$

Here, we suppressed the σ dependence of a_2 . We now write $(1 + \sigma \tilde{a}_2)^{-2} = 1 + \sigma \tilde{a}_2$ and for σ sufficiently small, \tilde{a}_2 satisfies all of the same bounds as a_2 because $|a_2| \lesssim 1$. Recalling the identities

$$(2.35) \quad \frac{d}{dx} \left\{ \frac{\operatorname{Ai}}{\operatorname{Bi}}(x) \right\} = -\pi^{-1} \operatorname{Bi}^{-2}(x), \quad \frac{d}{dx} \left\{ \frac{\operatorname{Bi}}{\operatorname{Ai}}(x) \right\} = \pi^{-1} \operatorname{Ai}^{-2}(x)$$

and the fact that $\operatorname{Ai}(u)$ and $\operatorname{Bi}(u)$ are strictly positive for $u \geq 0$, we integrate by parts to see that

$$(2.36) \quad \pi \int_{\tau}^{\infty} \operatorname{Bi}^{-2}(u)(1 + \sigma \tilde{a}_2(-\sigma^{\frac{2}{3}}u)) du = \left[\frac{\operatorname{Ai}}{\operatorname{Bi}}(u)(1 + \sigma \tilde{a}_2(-\sigma^{\frac{2}{3}}u)) \right] \Big|_{\infty}^{\tau} - \sigma^{\frac{5}{3}} \int_{\tau}^{\infty} \frac{\operatorname{Ai}}{\operatorname{Bi}}(u) \dot{\tilde{a}}_2(-\sigma^{\frac{2}{3}}u) du.$$

Therefore,

$$\phi_1(\zeta) = \operatorname{Ai}(\tau)(1 + \sigma a_2(\zeta)) \left[(1 + \sigma \tilde{a}_2(\zeta)) - \sigma^{\frac{5}{3}} \frac{\operatorname{Bi}}{\operatorname{Ai}}(\tau) \int_{\tau}^{\infty} \frac{\operatorname{Ai}}{\operatorname{Bi}}(u) \dot{\tilde{a}}_2(-\sigma^{\frac{2}{3}}u) du \right]$$

From this, we infer that

$$\begin{aligned} a_1(\zeta) &= a_2(\zeta) + (1 + \sigma a_2(\zeta)) \left[\tilde{a}_2(\zeta) + \sigma^{\frac{2}{3}} \frac{\operatorname{Bi}}{\operatorname{Ai}}(\tau) \int_{-\sigma^{-\frac{2}{3}}\zeta}^{\infty} \frac{\operatorname{Ai}}{\operatorname{Bi}}(u) \dot{\tilde{a}}_2(-\sigma^{\frac{2}{3}}u) du \right] \\ &:= a_2(\zeta) + (1 + \sigma a_2(\zeta)) [\tilde{a}_2(\zeta) + \tilde{a}_1(\zeta)] \end{aligned}$$

so it suffices to control

$$\tilde{a}_1(\zeta) = \sigma^{\frac{2}{3}} \frac{\operatorname{Bi}}{\operatorname{Ai}}(\tau) \int_{\tau}^{\infty} \frac{\operatorname{Ai}}{\operatorname{Bi}}(u) \dot{\tilde{a}}_2(-\sigma^{\frac{2}{3}}u) du.$$

To begin with, we use that $|\dot{\tilde{a}}_2(\zeta)| \lesssim \sigma^{-\frac{1}{3}}$ to write

$$\begin{aligned} |\tilde{a}_1(\zeta)| &\lesssim \sigma^{\frac{1}{3}} e^{\frac{4}{3} \langle \sigma^{-\frac{2}{3}}\zeta \rangle^{\frac{3}{2}}} \int_{-\sigma^{-\frac{2}{3}}\zeta}^{\infty} e^{-\frac{4}{3} \langle u \rangle^{\frac{3}{2}}} du \\ &\leq \sigma^{\frac{1}{3}} e^{\frac{4}{3} \langle \sigma^{-\frac{2}{3}}\zeta \rangle^{\frac{3}{2}}} \langle \sigma^{-\frac{2}{3}}\zeta \rangle^{-\frac{1}{2}} \int_{-\sigma^{-\frac{2}{3}}\zeta}^{\infty} e^{-\frac{4}{3} \langle u \rangle^{\frac{3}{2}}} \langle u \rangle^{\frac{1}{2}} du \\ &\lesssim \sigma^{\frac{1}{3}} \langle \sigma^{-\frac{2}{3}}\zeta \rangle^{-\frac{1}{2}}. \end{aligned}$$

We compute further that

$$(2.37) \quad \dot{\tilde{a}}_1(\zeta) = -\pi^{-1}\sigma^{-\frac{2}{3}}(\text{Ai Bi})^{-1}(\tau)\tilde{a}_1(\zeta) + \dot{\tilde{a}}_2(\zeta).$$

The second term we have already bounded by $\sigma^{-\frac{1}{3}}$ and the first obeys this bound as well because $|(\text{Ai Bi})^{-1}(x)| \lesssim \langle x \rangle^{\frac{1}{2}}$. Proceeding onward,

$$\partial_\zeta^2\{\tilde{a}_1(\zeta)\} = \pi^{-1}\sigma^{-\frac{4}{3}}\frac{d}{du} [(\text{Ai Bi})^{-1}(u)]|_{u=\tau}\tilde{a}_1(\zeta) - \pi^{-1}\sigma^{-\frac{2}{3}}(\text{Ai Bi})^{-1}(\tau)\dot{\tilde{a}}_1(\zeta) + \ddot{\tilde{a}}_2(\zeta).$$

From the fact that $|\frac{d}{du}\{(\text{Ai Bi})^{-1}(u)\}| \lesssim \langle u \rangle^{-\frac{1}{2}}$, it is easily checked as before that each term is bounded by at worst $\sigma^{-\frac{4}{3}}$, thus establishing all of the desired bounds on the ζ -derivatives of \tilde{a}_1 .

For the σ -derivatives, it is convenient to first rewrite

$$\tilde{a}_1(\sigma, \zeta) = \frac{\text{Bi}}{\text{Ai}}(\tau) \int_{-\zeta}^{\infty} \frac{\text{Ai}}{\text{Bi}}(\sigma^{-\frac{2}{3}}v) \dot{\tilde{a}}_2(\sigma, -v) dv,$$

so that

$$\begin{aligned} \partial_\sigma\{\tilde{a}_1(\sigma, \zeta)\} &= \frac{2}{3\pi}\sigma^{-\frac{5}{3}}\zeta(\text{Ai Bi})^{-1}(\tau)\tilde{a}_1(\sigma, \zeta) \\ &\quad + \frac{2}{3\pi}\sigma^{-\frac{5}{3}}\frac{\text{Bi}}{\text{Ai}}(\tau) \int_{-\zeta}^{\infty} \text{Bi}^{-2}(\sigma^{-\frac{2}{3}}v)v\dot{\tilde{a}}_2(\sigma, -v) dv \\ &\quad + \frac{\text{Bi}}{\text{Ai}}(\tau) \int_{-\zeta}^{\infty} \frac{\text{Ai}}{\text{Bi}}(\sigma^{-\frac{2}{3}}v)\partial_\sigma\{\dot{\tilde{a}}_2\}(\sigma, -v) dv \\ &=: D_1(\sigma, \zeta) + D_2(\sigma, \zeta) + D_3(\sigma, \zeta). \end{aligned}$$

Arguing as before, it is easy to see that $|D_1(\sigma, \zeta)| \lesssim \sigma^{-\frac{4}{3}}$ whereas

$$(2.38) \quad |D_2(\sigma, \zeta)| \lesssim \sigma^{-\frac{2}{3}}\frac{\text{Bi}}{\text{Ai}}(\tau) \int_{\tau}^{\infty} \text{Bi}^{-2}(v)v dv = \sigma^{-\frac{2}{3}}\frac{\text{Bi}}{\text{Ai}}(\tau) \left[\pi^{-1}\frac{\text{Ai}}{\text{Bi}}(\tau)\tau - \int_{\tau}^{\infty} \frac{\text{Ai}}{\text{Bi}}(u) du \right] \lesssim \sigma^{-\frac{4}{3}}$$

where the second term in brackets may be treated as the original estimate of \tilde{a}_1 . By using that $|\partial_\sigma\{\dot{\tilde{a}}_2(\sigma, \zeta)\}| \lesssim \sigma^{-2}$, we may similarly argue that $|D_3(\sigma, \zeta)| \lesssim \sigma^{-\frac{4}{3}}$. Thus, we conclude that $|\partial_\sigma\{\tilde{a}_1(\sigma, \zeta)\}| \lesssim \sigma^{-\frac{4}{3}}$.

For the mixed derivative $\partial_\sigma\{\dot{\tilde{a}}_1(\sigma, \zeta)\}$, we differentiate (2.37) to find that

$$\begin{aligned} \partial_\sigma\{\dot{\tilde{a}}_1(\sigma, \zeta)\} &= \frac{2}{3\pi}\sigma^{-\frac{5}{3}}(\text{Ai Bi})^{-1}(\tau)\tilde{a}_1(\sigma, \zeta) + \frac{2}{3\pi}\sigma^{-\frac{7}{3}}\zeta\frac{d}{du}\{(\text{Ai Bi})^{-1}(u)\}|_{u=\tau}\tilde{a}_1(\sigma, \zeta) \\ &\quad - \pi^{-1}\sigma^{-\frac{2}{3}}(\text{Ai Bi})^{-1}(\tau)\partial_\sigma\{\tilde{a}_1(\sigma, \zeta)\} + \partial_\sigma\{\dot{\tilde{a}}_2(\sigma, \zeta)\}, \end{aligned}$$

and it is merely a matter of collecting previously derived bounds to deduce that each term is bounded by σ^{-2} , except the third term is bounded by $\sigma^{-7/3}$.

Finally, to bound the second σ -derivative of \tilde{a}_2 , we comment on the derivatives of each of $D_i(\sigma, \zeta)$, $i = 1, 2, 3$ separately. The expression $\partial_\sigma\{D_1(\sigma, \zeta)\}$ is essentially the same as

$\partial_\sigma\{\dot{\tilde{a}}_2(\sigma, \zeta)\}$ with the loss of an additional σ power, so it is bounded in terms of σ^{-3} . For $D_2(\sigma, \zeta)$, one differentiates to find that

$$|\partial_\sigma\{D_2(\sigma, \zeta)\}| \lesssim \sigma^{-1}|D_2(\sigma, \zeta)| + \sigma^{-\frac{5}{3}}(\text{Ai Bi})^{-1}(\tau)|D_2(\sigma, \zeta)| + \sigma^{-\frac{10}{3}}\frac{\text{Bi}}{\text{Ai}}(\tau) \int_{-\zeta}^{\infty} \frac{d}{du}\{\text{Bi}^{-2}(u)\}|_{\sigma^{-\frac{2}{3}}v} v^2 dv + \sigma^{-4}\frac{\text{Bi}}{\text{Ai}}(\tau) \int_{-\zeta}^{\infty} \text{Bi}^{-2}(\sigma^{-\frac{2}{3}}v)v dv,$$

where in the second integral we have used the bound $|\dot{\tilde{a}}_2(\sigma, \zeta)| \lesssim \sigma^{-2}$. Collecting bounds easily shows that the first term is bounded by $\sigma^{-\frac{7}{3}}$ and the second by $\sigma^{-\frac{10}{3}}$. The first integral is bounded in terms of

$$\sigma^{-\frac{4}{3}}\frac{\text{Bi}}{\text{Ai}}(\tau) \int_{\tau}^{\infty} \frac{d}{du}\{\text{Bi}^{-2}(u)\}u^2 du = \sigma^{-\frac{4}{3}}\frac{\text{Bi}}{\text{Ai}}(\tau) \left[\text{Bi}^{-2}(\tau)\tau^2 - 2 \int_{\tau}^{\infty} \text{Bi}^{-2}(u)u du \right] \lesssim \sigma^{-\frac{8}{3}}$$

where we may bound the second term as in (2.38). Similarly, the second integral is bounded by $\sigma^{-10/3}$.

We now compute

$$\begin{aligned} \partial_\sigma\{D_3(\sigma, \zeta)\} &= \frac{2}{3\pi}\sigma^{-\frac{5}{3}}\zeta(\text{Ai Bi})^{-1}(\tau)D_3(\sigma, \zeta) + \frac{2}{3\pi}\sigma^{-\frac{5}{3}}\frac{\text{Bi}}{\text{Ai}}(\tau) \int_{-\zeta}^{\infty} \text{Bi}^{-2}(\sigma^{-\frac{2}{3}}v)v\partial_\sigma\{\dot{\tilde{a}}_2(\sigma, -v)\} dv \\ &\quad + \frac{\text{Bi}}{\text{Ai}}(\tau) \int_{-\zeta}^{\infty} \frac{\text{Ai}}{\text{Bi}}(\sigma^{-\frac{2}{3}}v)\partial_\sigma^2\{\dot{\tilde{a}}_2(\sigma, -v)\} dv. \end{aligned}$$

The first two terms are easily bounded by $\sigma^{-\frac{10}{3}}$. Using that $\partial_\sigma^2\{\dot{\tilde{a}}_2\}(\sigma, -v) = \frac{\partial}{\partial u}\{\partial_\sigma^2\{\tilde{a}_2(\sigma, u)\}\}|_{u=-v}$, we integrate by parts in the last term to rewrite it as

$$\frac{\text{Bi}}{\text{Ai}}(\tau) \left(\frac{\text{Ai}}{\text{Bi}}(\tau)\partial_\sigma^2\{\tilde{a}_2(\sigma, \zeta)\} - \pi^{-1}\sigma^{-\frac{2}{3}} \int_{-\zeta}^{\infty} \text{Bi}^{-2}(\sigma^{-\frac{2}{3}}v)\partial_\sigma^2\{\tilde{a}_2(\sigma, -v)\} dv \right),$$

which is controlled by $\partial_\sigma^2\{\tilde{a}_2(\sigma, \zeta)\} = O(\sigma^{-3})$. It follows then that $\partial_\sigma^2\{\dot{\tilde{a}}_2\} = O(\sigma^{-\frac{10}{3}})$. This was the last bound we needed to demonstrate, so the proof of the lemma is complete. \square

Corollary 2.9. *Let σ be sufficiently small. Then for $x \in [\frac{1}{2}, 1 + \delta]$,*

$$(2.39) \quad \begin{aligned} e(\sigma, r(x)) &= A(\sigma)(q(x))^{-\frac{1}{4}}\phi_1(\sigma, x) + B(\sigma)(q(x))^{-\frac{1}{4}}\phi_2(\sigma, x), \\ A(\sigma) &= \sigma^{-\frac{1}{6}}(2 + O(\sigma)), \quad B(\sigma) = O\left(\sigma^{\frac{5}{6}}e^{-\sigma^{-1}(\frac{\pi}{2}-1)}\right). \end{aligned}$$

Proof. We match the Bessel function approximation of $e(\sigma, r(x))$ to the basis $\{q^{-\frac{1}{4}}\phi_1(\tau), q^{-\frac{1}{4}}\phi_2(\tau)\}$ at $x = \frac{1}{2}$. We have that

$$(2.40) \quad e(\sigma, r(1/2)) = \sqrt{2}\sigma^{-\frac{1}{2}}[e^{\frac{\pi}{\sigma}} - 1]^{-\frac{1}{2}}\sqrt{\eta_*}I_1(\sigma^{-1}\eta_*)(1 + a(\sigma, \eta_*)),$$

where $\eta_* = \eta(\frac{1}{2})$ and we have used that $\eta'(\frac{1}{2}) = 1$ because $\eta' = \sqrt{Q}$. Using [8, (10.40.1)], we have from (2.40) that

$$e(\sigma, r(\frac{1}{2})) = 2\sqrt{2}[e^{\frac{\pi}{\sigma}} - 1]^{-\frac{1}{2}}\sqrt{\eta_*}\frac{e^{\frac{\eta_*}{\sigma}}\sigma^{\frac{1}{2}}}{\sqrt{2\pi\eta_*}}(1 + O(\sigma)) = \frac{2}{\sqrt{\pi}}e^{\frac{\eta_*}{\sigma}}[e^{\frac{\pi}{\sigma}} - 1]^{-\frac{1}{2}}(1 + O(\sigma))$$

$$= \frac{e^{\sigma^{-1}(\eta_* - \frac{\pi}{2})}}{\sqrt{\pi}}(2 + O(\sigma))$$

and also

$$\begin{aligned} \partial_x [e(\sigma, r(x))]_{x=\frac{1}{2}} &= 2\sqrt{2}[e^{\frac{\pi}{\sigma}} - 1]^{-\frac{1}{2}} \sqrt{\eta_*} \sigma^{-\frac{3}{2}} I_1'(\sigma^{-1}\eta_*)(1 + O(\sigma)) = \frac{2}{\sqrt{\pi}\sigma} e^{\frac{\eta_*}{\sigma}} [e^{\frac{\pi}{\sigma}} - 1]^{-\frac{1}{2}} (1 + O(\sigma)) \\ &= \frac{e^{\sigma^{-1}(\eta_* - \frac{\pi}{2})}}{\sqrt{\pi}\sigma} (2 + O(\sigma)). \end{aligned}$$

We also have that

$$\begin{aligned} \phi_1(\sigma, \zeta(\frac{1}{2})) &= q(\frac{1}{2})^{-\frac{1}{4}} \text{Ai}(\sigma^{-\frac{2}{3}}\zeta_*)(1 + a_1(\sigma, \zeta_*)) \\ \phi_2(\sigma, \zeta(\frac{1}{2})) &= q(\frac{1}{2})^{-\frac{1}{4}} \text{Bi}(\sigma^{-\frac{2}{3}}\zeta_*)(1 + a_2(\sigma, \zeta_*)) \end{aligned}$$

where $\zeta_* = -\zeta(\frac{1}{2})$. Because $\zeta_* > 0$, we use the asymptotics [8, (9.7.5) and (9.7.6)] to see that

$$\begin{aligned} \phi_1(\sigma, \zeta(x)) &= \sigma^{\frac{1}{6}} \frac{e^{-\frac{2\zeta_*^{\frac{3}{2}}}{3\sigma}}}{2\sqrt{\pi}} (1 + O(\sigma)) \\ \phi_2(\sigma, \zeta(x)) &= \sigma^{\frac{1}{6}} \frac{e^{\frac{2\zeta_*^{\frac{3}{2}}}{3\sigma}}}{\sqrt{\pi}} (1 + O(\sigma)) \end{aligned}$$

and [8, (9.7.6) and (9.7.8)] for

$$\begin{aligned} \partial_x [\phi_1(\sigma, \zeta(x))]_{x=\frac{1}{2}} &= -q^{-\frac{1}{4}}(\frac{1}{2}) \zeta'(\frac{1}{2}) \sigma^{-\frac{2}{3}} \text{Ai}'(\sigma^{-\frac{2}{3}}\zeta_*)(1 + O(\sigma)) = \sigma^{-\frac{5}{6}} \frac{e^{-\frac{2\zeta_*^{\frac{3}{2}}}{3\sigma}}}{2\sqrt{\pi}} (1 + O(\sigma)) \\ \partial_x [\phi_2(\sigma, \zeta(x))]_{x=\frac{1}{2}} &= -q^{-\frac{1}{4}}(\frac{1}{2}) \zeta'(\frac{1}{2}) \sigma^{-\frac{2}{3}} \text{Bi}'(\sigma^{-\frac{2}{3}}\zeta_*)(1 + O(\sigma)) = -\sigma^{-\frac{5}{6}} \frac{e^{\frac{2\zeta_*^{\frac{3}{2}}}{3\sigma}}}{\sqrt{\pi}} (1 + O(\sigma)) \end{aligned}$$

from using that $q(\frac{1}{2}) = \zeta_*^{-1} = [\zeta'(\frac{1}{2})]^2$. It follows that

$$W[\phi_1(\sigma, \zeta(\cdot)), \phi_2(\sigma, \zeta(\cdot))] = -\frac{\sigma^{-\frac{2}{3}}}{\pi} (1 + O(\sigma))$$

where W is the Wronskian evaluated at $x = \frac{1}{2}$ and also

$$\begin{aligned} W[e(\sigma, r(\cdot)), \phi_1(\sigma, \zeta(\cdot))] &= O\left(\sigma^{\frac{1}{6}} e^{\sigma^{-1}(\eta_* - \frac{\pi}{2} - \frac{2}{3}\zeta_*^{\frac{3}{2}})}\right) = O\left(\sigma^{\frac{1}{6}} e^{-\sigma^{-1}(\frac{\pi}{2}-1)}\right) \\ W[e(\sigma, r(\cdot)), \phi_2(\sigma, \zeta(\cdot))] &= -\frac{\sigma^{-\frac{5}{6}}}{\pi} e^{\sigma^{-1}(\eta_* - \frac{\pi}{2} + \frac{2}{3}\zeta_*^{\frac{3}{2}})} (2 + O(\sigma)) = -\frac{\sigma^{-\frac{5}{6}}}{\sqrt{\pi}} (2 + O(\sigma)) \end{aligned}$$

where the last equality on each line follows from the facts that $\eta_* - \frac{2}{3}\zeta_*^{\frac{3}{2}} = 1$ and $\eta_* + \frac{2}{3}\zeta_*^{\frac{3}{2}} = \frac{\pi}{2}$. Therefore,

$$A(\sigma) = \frac{W[e(\sigma, r(\cdot)), \phi_2(\sigma, \zeta(\cdot))]}{W[\phi_1(\sigma, \zeta(\cdot)), \phi_2(\sigma, \zeta(\cdot))]} = \sigma^{-\frac{1}{6}} (1 + O(\sigma)),$$

$$B(\sigma) = -\frac{W[e(\sigma, r(\cdot)), \phi_1(\sigma, \zeta(\cdot))]}{W[\phi_1(\sigma, \zeta(\cdot)), \phi_2(\sigma, \zeta(\cdot))]} = O\left(\sigma^{\frac{5}{6}} e^{-\sigma^{-1}(\frac{\pi}{2}-1)}\right).$$

□

2.3. Oscillatory Airy approximation: $x \gg 1$.

Proposition 2.10. *When $\zeta \geq 0$, the potential V satisfies the bounds*

$$|\partial_\zeta^j V(\zeta)| \lesssim \langle \zeta \rangle^{-2-j} \quad j = 0, 1, 2.$$

Proof. Since by the above ζ' is smooth and non-vanishing, the identity $q = (\zeta')^2$ shows that near 0, $V = -q^{-\frac{1}{4}} \frac{d^2 q^{\frac{1}{4}}}{d\zeta^2}$ is bounded, so we need only show that V has the claimed behavior as $\zeta \rightarrow \infty$. With $\cdot = \frac{d}{d\zeta}$, one computes first that

$$(2.41) \quad q^{-\frac{1}{4}} \frac{d^2 q^{\frac{1}{4}}}{d\zeta^2} = -\frac{3}{16} q^{-2} \dot{q}^2 + \frac{1}{4} q^{-1} \ddot{q}$$

so we need only find asymptotics for q in terms of ζ . Recalling definition of ζ as a function of x , we have that for $x \geq 1$

$$\frac{3}{2} \zeta^{\frac{3}{2}}(x) = \int_1^x \sqrt{1-u^{-1}} du = \int_1^x 1 + O(u^{-1}) du = x + c + O(x^{-1}).$$

The chain rule applied to the above equality then shows that $\zeta'(x) = O(\zeta^{-\frac{1}{2}})$, where every derivative of $O(\zeta^{-\frac{1}{2}})$ loses a power of ζ , that is, it exhibits symbol behavior. It follows then that $q = (\zeta')^2 = O(\zeta^{-1})$ for x large, from which the bound on V follows from (2.41) and one may obtain the bounds on V' and $V^{(2)}$ by differentiating this equality.

□

Proposition 2.11. *For any $\sigma > 0$ sufficiently small, the following holds: in the range $\zeta \geq 0$, a basis of solutions to (2.20) is given by*

$$(2.42) \quad \begin{aligned} \psi_+ &= (\text{Ai}(\tau) + i \text{Bi}(\tau))[1 + \sigma b_+(\sigma, \zeta)] \\ \psi_- &= (\text{Ai}(\tau) - i \text{Bi}(\tau))[1 + \sigma b_-(\sigma, \zeta)] \end{aligned}$$

with $\tau = -\sigma^{-\frac{2}{3}} \zeta$ and b_\pm are smooth functions that satisfy the bounds

$$\begin{aligned} |b_\pm(\sigma, \zeta)| &\lesssim \langle \zeta \rangle^{-\frac{3}{2}}, \quad |\dot{b}_\pm(\sigma, \zeta)| \lesssim \sigma^{-\frac{1}{3}} \left\langle \sigma^{-\frac{2}{3}} \zeta \right\rangle^{-\frac{1}{2}} \langle \zeta \rangle^{-2}, \quad |\ddot{b}_\pm(\sigma, \zeta)| \lesssim \sigma^{-1} \langle \zeta \rangle^{-2} \\ \partial_\sigma [b_\pm(\sigma, \zeta)] &\lesssim \sigma^{-1} \langle \zeta \rangle^{-\frac{3}{2}}, \quad \partial_\sigma^2 [b_\pm(\sigma, \zeta)] \lesssim \sigma^{-3}, \quad \partial_\sigma [\dot{b}_\pm(\sigma, \zeta)] \lesssim \sigma^{-2} \langle \zeta \rangle^{-1}. \end{aligned}$$

Remark 2.12. *This proposition is very similar to Proposition 9 of [7]. Indeed, the bounds on b_\pm and \dot{b}_\pm are produced via the same proof, as the only inputs are the asymptotics of V , which in this regime of ζ are the same as in that paper. However, for our purposes we also require an additional derivative in ζ and derivatives in the semi-classical parameter σ (\hbar in [7]). Note that*

in both settings, a representation of the form (2.42) is only possible because Ai and Bi have no common zeroes, and therefore (2.42) does not fix the zeroes of any solution.

Proof. Let $\psi_{\pm,0}(\zeta, \sigma) = \text{Ai}(\tau) \pm i \text{Bi}(\tau)$. Similar to (2.22), we obtain the equations

$$\left(\psi_{\pm,0}^2 \dot{b}_{\pm} \right)' = -\sigma^{-1} V \psi_{\pm,0}^2 (1 + \sigma b_{\pm})$$

whose solutions with $b_{\pm}(\infty) = 0$ and $\dot{b}_{\pm}(\infty) = 0$ is given by

$$(2.43) \quad b_{\pm}(\zeta) = -\sigma^{-1} \int_{\zeta}^{\infty} \int_{\zeta}^u \psi_{\pm,0}^{-2}(v) dv \psi_{\pm,0}^2(u) V(u) (1 + \sigma b_{\pm}(u)) du,$$

where for now we have suppressed the dependence on σ in the integrand. From [32], we have the asymptotic expansion

$$(2.44) \quad \text{Ai}(-z) \pm i \text{Bi}(-z) = \frac{1}{\sqrt{\pi} z^{\frac{1}{4}}} e^{\mp i \left(\frac{2}{3} z^{\frac{3}{2}} - \frac{\pi}{4} \right)} (1 + O(z^{-\frac{3}{2}}))$$

where the $O(z^{-\frac{3}{2}})$ term may be differentiated as a symbol. Thus, for $0 < x_0 < x_1$

$$(2.45) \quad \int_{x_0}^{x_1} (\text{Ai}(-z) \pm i \text{Bi}(-z))^{-2} dz \\ = \int_{x_0}^{x_1} z^{\frac{1}{2}} e^{\mp i \frac{4}{3} z^{\frac{3}{2}}} a(z) dz = \mp \frac{1}{2i} e^{\mp i \frac{4}{3} z^{\frac{3}{2}}} a(z) \Big|_{x_0}^{x_1} \pm \frac{1}{2i} \int_{x_0}^{x_1} e^{\mp i \frac{2}{3} z^{\frac{3}{2}}} a'(z) dz$$

for $a(z) = 1 + O(z^{-\frac{3}{2}})$. This shows that the above integral is $O(1)$ for all x_0 and x_1 . The main term with respect to σ in (2.43)

$$b_{\pm,0}(\zeta) = -\sigma^{-1} \int_{\zeta}^{\infty} \int_{\zeta}^u \psi_{\pm,0}^{-2}(v) dv \psi_{\pm,0}^2(u) V(u) du$$

satisfies the bound

$$|b_{\pm,0}(\zeta)| \lesssim \sigma^{\frac{1}{3}} \int_{\sigma^{-\frac{2}{3}} \zeta}^{\infty} \langle u \rangle^{-\frac{1}{2}} |V(-\sigma^{\frac{2}{3}} u)| du$$

where we have changed variables and used that by (2.44)

$$|(\text{Ai}(-z) + i \text{Bi}(-z))^2| \lesssim \langle z \rangle^{-\frac{1}{2}}$$

By Proposition 2.10, we see then that

$$|b_{\pm,0}(\zeta)| \lesssim \sigma^{\frac{1}{3}} \int_{\sigma^{-\frac{2}{3}} \zeta}^{\infty} \langle u \rangle^{-\frac{1}{2}} \left\langle \sigma^{\frac{2}{3}} u \right\rangle^{-2} du \lesssim \langle \zeta \rangle^{-\frac{3}{2}},$$

where the last inequality comes from bounding the integrand by $\sigma^{-\frac{4}{3}} u^{-\frac{5}{2}}$ when $\sigma^{-\frac{2}{3}} \zeta$ is large. We can now extend this bound to b_{\pm} by a contraction argument, as is explained in Proposition 2.2, by considering the linear operator

$$Ta = -\sigma^{-1} \int_{\zeta}^{\infty} \int_{\zeta}^u \psi_{\pm,0}^{-2}(v) dv \psi_{\pm,0}^2(u) V(u) a(u) du$$

as a map on the weighted space $\langle \zeta \rangle^{-\frac{3}{2}} L_{\zeta}^{\infty}$.

For the ζ -derivative bounds, we first write

$$(2.46) \quad \dot{b}_\pm(\zeta) = \sigma^{-1} \psi_{\pm,0}^{-2}(\zeta) \int_\zeta^\infty \psi_{\pm,0}^2(u) V(u) (1 + \sigma b_\pm(u)) du$$

and use (2.44) to see that $\psi_{\pm,0}^2(\zeta) = e^{i\frac{4}{3\sigma}\zeta^{\frac{3}{2}}} \omega(\sigma^{-\frac{2}{3}}\zeta)$ for some $\omega(u)$ with $|\omega(u)| \lesssim \langle u \rangle^{-\frac{1}{2}}$ and $|\omega'(u)| \lesssim \langle u \rangle^{-\frac{3}{2}}$. When $\zeta > \sigma^{-\frac{2}{3}}$, we may exploit the oscillatory phase by integrating by parts in the above integral via

$$\psi_{\pm,0}^2 = (2i\zeta^{\frac{1}{2}}/\sigma)^{-1} \omega(\sigma^{-\frac{2}{3}}\zeta) \frac{d}{d\zeta} [e^{i\frac{4}{3\sigma}\zeta^{\frac{3}{2}}}]$$

to find that

$$(2.47) \quad \dot{b}_\pm(\zeta) = \frac{1}{2i} \psi_{\pm,0}^{-2}(\zeta) [u^{-\frac{1}{2}} \omega(\sigma^{-\frac{2}{3}}u) e^{i\frac{4}{3\sigma}\zeta^{\frac{3}{2}}} V(u) (1 + \sigma b_\pm(u))] \Big|_\zeta^\infty$$

$$(2.48) \quad -\frac{1}{2i} \psi_{\pm,0}^{-2}(\zeta) \int_\zeta^\infty e^{i\frac{4}{3\sigma}\zeta^{\frac{3}{2}}} \frac{d}{du} [u^{-\frac{1}{2}} \omega(\sigma^{-\frac{2}{3}}u) V(u) (1 + \sigma b_\pm(u))] du.$$

Using the bounds on ω we see that the term on the right hand side of the equality in (2.47) is bounded by

$$\zeta^{-\frac{1}{2}} \langle \sigma^{-\frac{2}{3}}\zeta \rangle^{-\frac{1}{2}} \langle \zeta \rangle^{-2} \lesssim \sigma^{-\frac{1}{3}} \langle \sigma^{-\frac{2}{3}}\zeta \rangle^{-\frac{1}{2}} \langle \zeta \rangle^{-2},$$

with the last inequality coming from the assumption on ζ . By differentiating the product in the integrand of (2.48), one checks via the bounds on ω and Proposition 2.10 that every term other than the term in which the derivative falls on b_\pm is $O(\sigma^{-\frac{1}{3}} \langle \sigma^{-\frac{2}{3}}\zeta \rangle^{-\frac{1}{2}} \langle \zeta \rangle^{-2})$ so that we may write

$$\dot{b}_\pm(\zeta) = O(\sigma^{-\frac{1}{3}} \langle \sigma^{-\frac{2}{3}}\zeta \rangle^{-\frac{1}{2}} \langle \zeta \rangle^{-2}) - \frac{\sigma}{2i} \psi_{\pm,0}^{-2}(\zeta) \int_\zeta^\infty e^{i\frac{4}{3\sigma}\zeta^{\frac{3}{2}}} O(u^{-\frac{1}{2}} \langle \sigma^{-\frac{2}{3}}u \rangle^{-\frac{1}{2}} \langle u \rangle^{-2}) \dot{b}_\pm(u) du.$$

By iterating this equality, we see that the second term is better than the first one, so we see that $|\dot{b}_\pm(\zeta)| \lesssim \sigma^{-\frac{1}{3}} \langle \sigma^{-\frac{2}{3}}\zeta \rangle^{-\frac{1}{2}} \langle \zeta \rangle^{-2}$. As for the case when $\sigma^{-\frac{2}{3}}\zeta \leq 1$, we simply write (2.46) as

$$\begin{aligned} & \sigma^{-1} \psi_{\pm,0}^{-2}(\zeta) \int_\zeta^{\sigma^{\frac{2}{3}}} \psi_{\pm,0}^2(u) V(u) (1 + \sigma b_\pm(u)) du \\ & + \sigma^{-1} \psi_{\pm,0}^{-2}(\zeta) \int_{\sigma^{\frac{2}{3}}}^\infty \psi_{\pm,0}^2(u) V(u) (1 + \sigma b_\pm(u)) du, \end{aligned}$$

where the first term is clearly bounded by $\sigma^{-\frac{1}{3}}$ and the second one can be estimated similar to (2.47). Thus, in either case we see that

$$|\dot{b}_\pm(\zeta)| \lesssim \sigma^{-\frac{1}{3}} \langle \sigma^{-\frac{2}{3}}\zeta \rangle^{-\frac{1}{2}} \langle \zeta \rangle^{-2}$$

as claimed.

For the second ζ -derivative, we differentiate (2.46) to find that

$$\ddot{b}_{\pm}(\zeta) = -\sigma^{-1}V(\zeta)(1 + \sigma b_{\pm}(\zeta)) - 2\dot{\psi}_{\pm,0}(\zeta)\psi_{\pm,0}^{-1}(\zeta)\dot{b}_{\pm}(\zeta)$$

Since $|\psi_{\pm,0}^{-1}(\zeta)| \lesssim \langle \sigma^{-\frac{2}{3}}\zeta \rangle^{\frac{1}{4}}$ and by (2.44) $|\dot{\psi}_{\pm,0}(\zeta)| \lesssim \sigma^{-\frac{2}{3}} \langle \sigma^{-\frac{2}{3}}\zeta \rangle^{\frac{1}{4}}$, the previously derived bounds on b_{\pm} and \dot{b}_{\pm} show that

$$|\ddot{b}_{\pm}(\zeta)| \lesssim \sigma^{-1} \langle \zeta \rangle^{-2}$$

We now demonstrate the bounds on the σ -derivatives of b_{\pm} . To begin, we rewrite (2.43) as

$$\begin{aligned} b_{\pm}(\sigma, \zeta) &= -\sigma^{\frac{1}{3}} \int_{\sigma^{-\frac{2}{3}}\zeta}^{\infty} \int_{\sigma^{-\frac{2}{3}}\zeta}^u (\text{Ai} \pm i \text{Bi})^{-2}(-v) dv (\text{Ai} \pm i \text{Bi})^2(-u) V(\sigma^{\frac{2}{3}}u) (1 + \sigma b(\sigma, \sigma^{\frac{2}{3}}u)) du \\ &= -\sigma^{\frac{1}{3}} \int_{\sigma^{-\frac{2}{3}}\zeta}^{\infty} \int_{\sigma^{-\frac{2}{3}}\zeta}^u (\text{Ai} \pm i \text{Bi})^{-2}(-v) dv (\text{Ai} \pm i \text{Bi})^2(-u) F(u, \sigma) du \end{aligned}$$

and then differentiate with respect to σ to find that

$$\begin{aligned} \partial_{\sigma}[b_{\pm}(\sigma, \zeta)] &= \frac{1}{3}\sigma^{-1}b_{\pm}(\sigma, \zeta) \\ &\quad - \frac{2}{3}\sigma^{-\frac{4}{3}}\psi_{\pm,0}^{-2}(\zeta)\zeta \int_{\sigma^{-\frac{2}{3}}\zeta}^{\infty} (\text{Ai} \pm i \text{Bi})^2(-u) F(u, \sigma) du \\ &\quad - \sigma^{\frac{1}{3}} \int_{\sigma^{-\frac{2}{3}}\zeta}^{\infty} \int_{\sigma^{-\frac{2}{3}}\zeta}^u (\text{Ai} \pm i \text{Bi})^{-2}(-v) dv (\text{Ai} \pm i \text{Bi})^2(-u) \partial_{\sigma}[F(u, \sigma)] du \\ &=: F_1^{\pm}(\sigma, \zeta) + F_2^{\pm}(\sigma, \zeta) + F_3^{\pm}(\sigma, \zeta). \end{aligned}$$

Our previous bound on b shows that $|F_1^{\pm}(\sigma, \zeta)| \lesssim \sigma^{-1} \langle \zeta \rangle^{-\frac{3}{2}}$. Also, from (2.46), we see that $F_2^{\pm}(\sigma, \zeta) = -\frac{2}{3}\sigma^{-1}\zeta\dot{b}_{\pm}(\sigma, \zeta)$ and is therefore bounded by $\sigma^{-\frac{4}{3}} \langle \sigma^{-\frac{2}{3}}\zeta \rangle^{-\frac{1}{2}} \langle \zeta \rangle^{-2} \zeta$, which is again bounded by $\sigma^{-1} \langle \zeta \rangle^{-\frac{3}{2}}$. We compute that

$$\begin{aligned} (2.49) \quad \partial_{\sigma}[F(u, \sigma)] &= \frac{2}{3}\sigma^{-\frac{1}{3}}V'(\sigma^{\frac{2}{3}}u)u(1 + \sigma b_{\pm}(\sigma, \sigma^{\frac{2}{3}}u)) + V(\sigma^{\frac{2}{3}}u)b_{\pm}(\sigma, \sigma^{\frac{2}{3}}u) \\ &\quad + \frac{2}{3}\sigma^{\frac{2}{3}}V(\sigma^{\frac{2}{3}}u)\dot{b}_{\pm}(\sigma, \sigma^{\frac{2}{3}}u) + \sigma V(\sigma^{\frac{2}{3}}u)\partial_{\sigma}[b_{\pm}(\sigma, v)]_{v=\sigma^{\frac{2}{3}}u} \\ &= O\left(\sigma^{-\frac{1}{3}}\langle u \rangle \langle \sigma^{\frac{2}{3}}u \rangle^{-3}\right) + O\left(\langle \sigma^{\frac{2}{3}}u \rangle^{-\frac{7}{2}}\right) + O\left(\sigma^{\frac{1}{3}}\langle u \rangle^{-\frac{1}{2}} \langle \sigma^{\frac{2}{3}}u \rangle^{-4}\right) \\ &\quad + \sigma V(\sigma^{\frac{2}{3}}u)\partial_{\sigma}[b_{\pm}(\sigma, v)]_{v=\sigma^{\frac{2}{3}}u} \\ &= O\left(\sigma^{-\frac{1}{3}}\langle u \rangle \langle \sigma^{\frac{2}{3}}u \rangle^{-3}\right) + \sigma V(\sigma^{\frac{2}{3}}u)\partial_{\sigma}[b_{\pm}(\sigma, v)]_{v=\sigma^{\frac{2}{3}}u}. \end{aligned}$$

Inserting the last line into $F_3^\pm(\sigma, \zeta)$ shows that

$$\begin{aligned} F_3^\pm(\sigma, \zeta) &= \int_{\sigma^{-\frac{2}{3}}\zeta}^{\infty} O\left(\langle u \rangle^{\frac{1}{2}} \langle \sigma^{\frac{2}{3}}u \rangle^{-3}\right) du + T\left(\partial_\sigma[b_\pm(\sigma, v)]_{v=\sigma^{\frac{2}{3}}u}\right) \\ &= O\left(\sigma^{-1} \langle \zeta \rangle^{-\frac{3}{2}}\right) + T\left(\partial_\sigma[b_\pm(\sigma, v)]_{v=\sigma^{\frac{2}{3}}u}\right) \end{aligned}$$

with

$$T(a) = -\sigma^{\frac{4}{3}} \int_{\sigma^{-\frac{2}{3}}\zeta}^{\infty} \int_{\sigma^{-\frac{2}{3}}\zeta}^u (\text{Ai} \pm i \text{Bi})^{-2}(-v) dv (\text{Ai} \pm i \text{Bi})^2(-u) V(\sigma^{\frac{2}{3}}u) a(\sigma^{\frac{2}{3}}u) du.$$

Collecting various bounds, we have shown that

$$(2.50) \quad \partial_\sigma[b_\pm(\sigma, \zeta)] = O\left(\sigma^{-1} \langle \zeta \rangle^{-\frac{3}{2}}\right) + T\left(\partial_\sigma[b_\pm(\sigma, v)]_{v=\sigma^{\frac{2}{3}}u}\right).$$

For small enough σ , T is a contraction on the weighted space $\langle \zeta \rangle^{\frac{3}{2}} L_\zeta^\infty$ because

$$|T(\langle \zeta \rangle^{-\frac{3}{2}})| \lesssim \sigma^{\frac{4}{3}} \int_{\sigma^{-\frac{2}{3}}\zeta}^{\infty} \langle u \rangle^{-\frac{1}{2}} \langle \sigma^{\frac{2}{3}}u \rangle^{-\frac{7}{2}} du \lesssim \sigma \langle \zeta \rangle^{-3}$$

so we may conclude that the first term in (2.50) bounds the second, i.e.

$$|\partial_\sigma[b_\pm(\sigma, \zeta)]| \lesssim \sigma^{-1} \langle \zeta \rangle^{-\frac{3}{2}}.$$

The second σ -derivative will require an estimate on the mixed derivative $\partial_\sigma[\dot{b}(\sigma, \zeta)]$. This easily follows from the bounds we have in hand by differentiating in ζ each of $F_i^\pm(\sigma, \zeta)$ for $i = 1, 2, 3$. Clearly $\partial_\zeta[F_1^\pm(\sigma, \zeta)]$ contributes $\sigma^{-\frac{4}{3}} \langle \sigma^{-\frac{2}{3}}\zeta \rangle^{-\frac{1}{2}} \langle \zeta \rangle^{-2}$ while it is easy to see that

$$\begin{aligned} \partial_\zeta[F_2^\pm(\sigma, \zeta)] &= -2\psi_{\pm,0}^{-1}(\zeta)\dot{\psi}_{\pm,0}(\zeta)F_2^\pm(\sigma, \zeta) + \zeta^{-1}F_2^\pm(\sigma, \zeta) + \frac{2}{3}\sigma^{-2}\zeta V(\zeta)(1 + \sigma b_\pm(\sigma, \zeta)) \\ &= O\left(\sigma^{-2} \langle \zeta \rangle^{-1}\right). \end{aligned}$$

Finally, the bound we have obtained on $\partial_\sigma[b(\sigma, \zeta)]$ may be used in (2.49) to show that

$$\begin{aligned} \partial_\zeta[F_3^\pm(\sigma, \zeta)] &= \sigma^{-\frac{1}{3}}\psi_{\pm,0}^{-2}(\zeta) \int_{\sigma^{-\frac{2}{3}}\zeta}^{\infty} (\text{Ai} \pm i \text{Bi})^2(-u)\partial_\sigma[F(u, \sigma)] du \\ &\lesssim \sigma^{-\frac{2}{3}} \langle \sigma^{-\frac{2}{3}}\zeta \rangle^{\frac{1}{2}} \int_{\sigma^{-\frac{2}{3}}\zeta}^{\infty} \langle u \rangle^{\frac{1}{2}} \langle \sigma^{\frac{2}{3}}u \rangle^{-3} du \lesssim \sigma^{-2} \langle \zeta \rangle^{-1} \end{aligned}$$

for a total bound of

$$\partial_\sigma[\dot{b}(\sigma, \zeta)] \lesssim \sigma^{-2} \langle \zeta \rangle^{-1}.$$

For the second σ -derivative, we proceed similarly and differentiate each of $F_i^\pm(\sigma, \zeta)$. First,

$$\partial_\sigma[F_1^\pm(\sigma, \zeta)] = -\sigma^{-1}(F_1^\pm(\sigma, \zeta)) + \frac{1}{3}\sigma^{-1}\partial_\sigma[b_\pm(\sigma, \zeta)] \lesssim \sigma^{-2} \langle \zeta \rangle^{-\frac{3}{2}}$$

whereas

$$\begin{aligned} \partial_\sigma[F_2^\pm(\sigma, \zeta)] &= -\frac{4}{3}\sigma^{-1}F_2^\pm(\sigma, \zeta) - \frac{4}{3}\sigma^{-\frac{5}{3}}\zeta\psi_{\pm,0}^{-1}(\zeta)(\text{Ai} \pm i \text{Bi})'(-\sigma^{-\frac{2}{3}}\zeta)F_2^\pm(\sigma, \zeta) \\ &\quad - \frac{4}{9}\sigma^{-3}\zeta^2V(\zeta)(1 + \sigma b_\pm(\sigma, \zeta)) - \frac{2}{3}\sigma^{-\frac{4}{3}}\psi_{\pm,0}^{-2}(\zeta)\zeta \int_{\sigma^{-\frac{2}{3}}\zeta}^{\infty} (\text{Ai} \pm \text{Bi})^2(-u)\partial_\sigma\{F(u, \sigma)\} du. \end{aligned}$$

The first term is $O(\sigma^{-2} \langle \zeta \rangle^{-\frac{3}{2}})$ and second term is $O(\sigma^{-3})$ because we have already shown that $|F_2^\pm(\sigma, \zeta)| \lesssim \sigma^{-\frac{4}{3}} \langle \sigma^{-\frac{2}{3}}\zeta \rangle^{-\frac{1}{2}} \langle \zeta \rangle^{-2} \zeta$. The third term is also $O(\sigma^{-3})$ and the fourth is bounded by $\sigma^{-\frac{5}{3}}\psi_{\pm,0}^{-2}(\zeta)\zeta F_3^\pm(\sigma, \zeta)$ which is again $O(\sigma^{-3})$ by the previously obtained bound on $F_3^\pm(\sigma, \zeta)$.

We compute further that

$$\begin{aligned} \partial_\sigma[F_3^\pm(\sigma, \zeta)] &= \frac{1}{3}\sigma^{-1}F_3^\pm(\sigma, \zeta) - \frac{2}{3}\sigma^{-\frac{4}{3}}\psi_{\pm,0}^{-2}(\zeta) \int_{\sigma^{-\frac{2}{3}}\zeta}^{\infty} (\text{Ai} \pm i \text{Bi})^2(-u)\partial_\sigma[F(u, \sigma)] du \\ &\quad - \sigma^{\frac{1}{3}} \int_{\sigma^{-\frac{2}{3}}\zeta}^{\infty} \int_{\sigma^{-\frac{2}{3}}\zeta}^u (\text{Ai} \pm i \text{Bi})^{-2}(-v) dv (\text{Ai} \pm i \text{Bi})^2(-u)\partial_\sigma^2[F(u, \sigma)] du. \end{aligned}$$

Arguing similarly, one may easily see that the first two terms are $O(\sigma^{-3})$. For the last term, we need to estimate $\partial_\sigma^2[F(u, \sigma)]$. Series of elementary operations show that

$$\partial_\sigma^2[F(u, \sigma)] = O\left(\sigma^{-\frac{4}{3}} \langle \sigma^{\frac{2}{3}}u \rangle^{-2} \langle u \rangle\right) + \sigma V(\sigma^{\frac{2}{3}}u)\partial_\sigma^2[b_\pm(\sigma, v)]_{v=\sigma^{\frac{2}{3}}u}$$

and therefore

$$\begin{aligned} \partial_\sigma[F_3^\pm(\sigma, \zeta)] &= O(\sigma^{-3}) \\ &\quad - \sigma^{\frac{1}{3}} \int_{\sigma^{-\frac{2}{3}}\zeta}^{\infty} \int_{\sigma^{-\frac{2}{3}}\zeta}^u (\text{Ai} \pm i \text{Bi})^{-2}(-v) dv (\text{Ai} \pm i \text{Bi})^2(-u) O\left(\sigma^{-\frac{4}{3}} \langle \sigma^{\frac{2}{3}} \rangle^{-2} \langle u \rangle\right) du \\ &\quad + T\left(\partial_\sigma^2[b_\pm(\sigma, v)]_{v=\sigma^{\frac{2}{3}}u}\right). \end{aligned}$$

The middle term is easily seen to be $O(\sigma^{-2} \langle \zeta \rangle^{-\frac{1}{2}})$ so that all of our estimates on the σ -derivatives of $F_i^\pm(\sigma, \zeta)$ show that $\partial_\sigma^2[b_\pm(\sigma, \zeta)]$ satisfies a fixed point equation of the form

$$\partial_\sigma^2[b_\pm(\sigma, \zeta)] = O(\sigma^{-3}) + T\left(\partial_\sigma^2[b_\pm(\sigma, v)]_{v=\sigma^{\frac{2}{3}}u}\right).$$

To conclude, we need only show that T is a contraction on L^∞ for σ sufficiently small, but this follows from the computation

$$|T(1)| \lesssim \sigma^{\frac{4}{3}} \int_{\sigma^{-\frac{2}{3}}}^{\infty} \langle u \rangle^{-\frac{1}{2}} \left\langle \sigma^{\frac{2}{3}} u \right\rangle^{-2} du \lesssim \sigma \langle \zeta \rangle^{-\frac{3}{2}}$$

so that as before it follows that $\partial_\sigma^2 [b_\pm(\sigma, \zeta)] = O(\sigma^{-3})$, which completes the proof. \square

We would now like to use the oscillatory basis to provide an approximation of $e(\sigma, r)$ that is suitable for use inside the oscillatory integral defining K_t . This requires detailed bounds on the function

$$\zeta_r(\sigma) := \sigma^{-1} \frac{2}{3} \zeta^{\frac{3}{2}}(\sigma^2 r).$$

Proposition 2.13. *Fix constants $k > 2$, $0 < c < 1$, and $s > kc^{-2}$. Then for any $r \geq s$ the function $\zeta_r(\sigma) = \sigma^{-1} \frac{2}{3} \zeta^{\frac{3}{2}}(\sigma^2 r)$ satisfies the inequalities*

$$(2.51) \quad \zeta_r'(\sigma) \sim r$$

$$(2.52) \quad |\zeta_r''(\sigma)| \lesssim \frac{r}{\sigma}$$

$$(2.53) \quad \zeta_r''(\sigma) < 0$$

uniformly for σ in the region $[ks^{-\frac{1}{2}}, c]$. Furthermore, for if $s < r$ then

$$(2.54) \quad r - s \lesssim \zeta_r'(\sigma) - \zeta_s'(\sigma) \lesssim r$$

$$(2.55) \quad |\zeta_r''(\sigma) - \zeta_s''(\sigma)| \lesssim \frac{r - s}{\sigma}$$

$$(2.56) \quad \zeta_r''(\sigma) - \zeta_s''(\sigma) < 0$$

Here, all derivatives are with respect to σ and \lesssim and \sim indicate bounds with respect to constants that depends only on k and c (i.e. not on r and s).

Proof. Recall that for $z \geq 1$

$$\frac{2}{3} \zeta^{\frac{3}{2}}(z) = \int_1^z \sqrt{1 - u^{-1}} du$$

or explicitly

$$(2.57) \quad \frac{2}{3} \zeta^{\frac{3}{2}}(z) = \sqrt{z(z-1)} - \log(\sqrt{z} + \sqrt{z-1})$$

One computes that

$$\begin{aligned} \zeta_r'(\sigma) &= 2r \sqrt{1 - (\sigma^2 r)^{-1}} - \frac{2}{3} \zeta^{\frac{3}{2}}(\sigma^2 r) \sigma^{-2} \\ &= r \sqrt{1 - (\sigma^2 r)^{-1}} + \sigma^{-2} \log(\sigma r^{\frac{1}{2}} + \sqrt{\sigma^2 r - 1}). \end{aligned}$$

Since $\sigma \gtrsim s^{-\frac{1}{2}}$ on the regime in question, the log term is positive so the above is clearly greater than $r\sqrt{1-k^{-2}}$. For the upper bound, one writes

$$\log(\sigma r^{\frac{1}{2}} + \sqrt{\sigma^2 r - 1}) \leq \log(2\sigma r^{\frac{1}{2}})$$

and then checks that as a function of σ , $\sigma^{-2} \log(\sigma a)$ has a global maximum of $\frac{a^2}{2e}$.

To bound the second derivative, we first calculate

$$\begin{aligned} \zeta_r''(\sigma) &= 2\sigma^{-3}(1 - (\sigma^2 r)^{-1})^{-\frac{1}{2}} - 2r\sigma^{-1}\sqrt{1 - (\sigma^2 r)^{-1}} + 2\sigma^{-3}\frac{2}{3}\zeta^{\frac{3}{2}}(\sigma^2 r) \\ &= 2\sigma^{-3}(1 - (\sigma^2 r)^{-1})^{-\frac{1}{2}} - 2\sigma^{-3}\log(\sigma r^{\frac{1}{2}} + \sqrt{\sigma^2 r - 1}) \end{aligned}$$

and then observe that because $\sigma^{-2} \lesssim r$

$$|\zeta_r''(\sigma)| \lesssim \sigma^{-1}(r + \sigma^{-2}\log(2\sigma r^{\frac{1}{2}})) \lesssim \frac{r}{\sigma}.$$

The negativity of ζ_r'' follows from the above expression and the fact that $(1 - (\sigma^2 r)^{-1})^{-\frac{1}{2}}$ is a decreasing function of $\sigma^2 r$ while $\log(\sigma r^{\frac{1}{2}} + \sqrt{\sigma^2 r - 1})$ is an increasing function of $\sigma^2 r$ and one may verify that their difference is negative at, say, $\sigma^2 r = 2$.

For the estimate on the difference $\zeta_r' - \zeta_s'$, we observe first that $\zeta_r(\sigma)$ is an increasing function of r for any fixed σ and $\frac{d\zeta_r'}{dr}$ is uniformly bounded below and above for all allowed σ and s . Hence, the mean value theorem implies (2.54) and (2.55). To see (2.56), notice that ζ_r'' is a negative decreasing function of r for any fixed σ . \square

Corollary 2.14. *Let $\sigma < c$ for $c > 0$ sufficiently small. Then for all $x \geq 1$*

$$(2.58) \quad \begin{aligned} e(\sigma, r(x)) &= c_-(\sigma)q^{-\frac{1}{4}}\psi_+(\tau(x)) + c_+(\sigma)q^{-\frac{1}{4}}\psi_-(\tau(x)) \\ c_-(\sigma) &= \sigma^{-\frac{1}{6}}(1 + O(\sigma)), \quad c_+(\sigma) = \sigma^{-\frac{1}{6}}(1 + O(\sigma)). \end{aligned}$$

Furthermore, for some constant $C > 0$ large enough, we may write

$$e(\sigma, r) = e^{i\zeta_r(\sigma)}a_+(\sigma, r) + e^{-i\zeta_r(\sigma)}a_-(\sigma, r)$$

where the functions a_{\pm} are smooth and satisfy the bounds

$$\begin{aligned} |a_{\pm}(\sigma, r)| &\lesssim 1 \\ |\partial_{\sigma}[a_{\pm}(\sigma, r)]| &\lesssim \sigma^{-1} \lesssim s^{\frac{1}{2}} \\ |\partial_{\sigma}^2[a_{\pm}(\sigma, r)]| &\lesssim s \end{aligned}$$

uniformly for all $0 < s \leq r$ and $\sigma \in [Cs^{-\frac{1}{2}}, c]$.

Proof. First, we connect the expression derived for $e(\sigma, r)$ in Corollary 2.9 to the basis constructed in Proposition 2.11 at $\zeta = 0$ (that is, $x = 1$). Based on Corollary 2.9, by regarding r as a function $x(\zeta)$ we have

$$e(\sigma, r)|_{\zeta=0} = 2\sigma^{-\frac{1}{6}} \text{Ai}(0)(1 + O(\sigma))$$

$$\partial_\zeta [e(\sigma, r)]|_{\zeta=0} = -2\sigma^{-\frac{5}{6}} \text{Ai}'(0)(1 + O(\sigma))$$

since we may absorb the Bi term into the $O(\sigma)$ because the coefficient $B(\sigma) = O(\sigma^\infty)$ and $q(0) = 1$. Now, all of the Wronskians are evaluated at $\zeta = 0$. It follows that

$$W[e(\sigma, r(x(\cdot))), \psi_\pm(\sigma, \cdot)] = \mp 2i\sigma^{-\frac{5}{6}} W[\text{Ai}, \text{Bi}](1 + O(\sigma)) = \mp \frac{i}{2\pi} \sigma^{-\frac{5}{6}} (1 + O(\sigma))$$

and because

$$\begin{aligned} W[\psi_+(\sigma, \cdot), \psi_-(\sigma, \cdot)] &= -\sigma^{-\frac{2}{3}} W[\text{Ai} + i\text{Bi}, \text{Ai} - i\text{Bi}](1 + O(\sigma)) = 2i\sigma^{-\frac{2}{3}} W[\text{Ai}, \text{Bi}](1 + O(\sigma)) \\ &= \frac{2i}{\pi} \sigma^{-\frac{2}{3}} (1 + O(\sigma)) \end{aligned}$$

from which we may determine $c_\pm(\sigma)$ immediately. Now, using (2.44), one may easily see that

$$\begin{aligned} e(\sigma, r) &= (-Q)^{-\frac{1}{4}}(x) e^{-i(\zeta_r(\sigma) - \frac{\pi}{4})} (1 + O((\zeta_r(\sigma))^{-1}) (1 + \sigma b_+(\sigma)) \\ &\quad + (-Q)^{-\frac{1}{4}}(x) e^{i(\zeta_r(\sigma) - \frac{\pi}{4})} (1 + O((\zeta_r(\sigma))^{-1}) (1 + \sigma b_-(\sigma)) \\ &= e^{-i\zeta_r(\sigma)} a_-(\sigma, r) + e^{i\zeta_r(\sigma)} a_+(\sigma, r) \end{aligned}$$

so we are only left to show the bounds on a_\pm under the assumption that $r \geq s > C\sigma^{-2}$. The bound $|a_\pm| \lesssim 1$ is immediate from the fact that for $\sigma^2 r > C$, $\zeta_r(\sigma)$ is bounded below. For the derivatives, by making liberal use of the fact that $\sigma^2 r > C$, one computes that

$$\begin{aligned} |(-Q(\sigma^2 r))^{-\frac{1}{4}}| &\lesssim 1 \\ |\partial_\sigma [(-Q(\sigma^2 r))^{-\frac{1}{4}}]| &= \frac{1}{2} (1 - (\sigma^2 r)^{-1})^{-\frac{5}{4}} \sigma^{-3} r \lesssim \sigma^{-1} \lesssim s^{\frac{1}{2}} \\ |\partial_\sigma^2 [(-Q(\sigma^2 r))^{-\frac{1}{4}}]| &= \frac{5}{4} (1 - (\sigma^2 r)^{-1})^{-\frac{9}{4}} \sigma^{-6} r^{-2} + \frac{3}{2} (1 - (\sigma^2 r)^{-1})^{-\frac{5}{4}} \sigma^{-4} r^{-1} \lesssim \sigma^{-2} \lesssim s. \end{aligned}$$

Taylor expanding (2.57) we also have $\zeta_r(\sigma) \sim \sigma r$ when $\sigma^2 r > 2$. Therefore, by (2.51) we obtain

$$|\partial_\sigma [(\zeta_r(\sigma))^{-1}]| = |\zeta_r'(\sigma) (\zeta_r(\sigma))^{-2}| \lesssim 1.$$

Similarly

$$|\partial_\sigma^2 [(\zeta_r(\sigma))^{-1}]| \lesssim |\zeta_r''(\sigma) (\zeta_r(\sigma))^{-2}| + |(\zeta_r'(\sigma))^2 (\zeta_r(\sigma))^{-3}| \lesssim \sigma^{-1} \lesssim s^{\frac{1}{2}}$$

again from Proposition 2.13. Now, one uses Proposition 2.11 and Proposition 2.13 to see that

$$\begin{aligned} |b_\pm(\sigma, \zeta(\sigma^2 r))| &\lesssim 1 \\ |\partial_\sigma [b_\pm(\sigma, \zeta(\sigma^2 r))]| &\lesssim |\sigma r \zeta'(\sigma^2 r) \dot{b}_\pm(\sigma, \zeta(\sigma^2 r))| + \partial_\sigma [b_\pm(\sigma, \zeta)]|_{\zeta=\zeta(\sigma^2 r)} \\ &\lesssim (\sigma r) (\sigma^2 r)^{-\frac{1}{3}} (\zeta(\sigma^2 r))^{-\frac{5}{2}} + \sigma^{-1} \lesssim \sigma^{-1} \lesssim s^{\frac{1}{2}}, \end{aligned}$$

where $\dot{\cdot}$ represents the derivative with respect to the second variable and thus

$$\begin{aligned} |1 + \sigma b_\pm(\sigma, \zeta(\sigma^2 r))| &\lesssim 1 \\ |\partial_\sigma [\sigma b_\pm(\sigma, \zeta(\sigma^2 r))]| &\lesssim |b_\pm(\sigma, \zeta(\sigma^2))| + |\sigma^2 r \zeta'(\sigma^2 r) \dot{b}_\pm(\sigma, \zeta(\sigma^2 r))| + \sigma \partial_\sigma [b_\pm(\sigma, \zeta)]|_{\zeta=\zeta(\sigma^2 r)} \end{aligned}$$

$$\lesssim (\zeta(\sigma^2 r))^{-\frac{3}{2}} + (\sigma^2 r)(\sigma^2 r)^{-\frac{1}{3}}(\zeta(\sigma^2 r))^{-\frac{5}{2}} + (\zeta(\sigma^2 r))^{-\frac{3}{2}} \lesssim 1.$$

As usual suppressing the variable $\sigma^2 r$ in ζ , we obtain

$$|\partial_\sigma^2[\sigma b_\pm(\sigma, \zeta)]| \lesssim |\partial_\sigma[b_\pm(\sigma, \zeta(\sigma^2 r))]| + \sigma|\partial_\sigma^2[b_\pm(\sigma, \zeta(\sigma^2 r))]|.$$

The first term is less than σ^{-1} by our previous computation and for the second we write

$$\begin{aligned} \partial_\sigma^2[b_\pm(\sigma, \zeta(\sigma^2 r))] &= \partial_\sigma^2[b_\pm(\sigma, \zeta(x))]_{x=\sigma^2 r} + (2r\zeta'(\sigma^2 r) + 4(\sigma r)^2\zeta''(\sigma^2 r))\dot{b}_\pm(\sigma, \zeta) \\ &\quad + 2\sigma r\zeta'(\sigma^2 r)\partial_\sigma[\dot{b}_\pm(\sigma, \zeta(x))]_{x=\sigma^2 r} + (2\sigma r\zeta'(\sigma^2 r))^2\ddot{b}_\pm(\sigma, \zeta) \end{aligned}$$

and using various bounds from Proposition 2.11 shows that $|\partial_\sigma^2[b_\pm(\sigma, \zeta(x))]_{x=\sigma^2 r}| \lesssim \sigma^{-3}$.

This gives a total bound of

$$|\partial_\sigma^2[b_\pm(\sigma, \zeta(\sigma^2 r))]| \lesssim \sigma^{-2} \lesssim s.$$

Summarizing all of these derivative computations, we have shown that

$$\begin{aligned} |(-Q(\sigma^2 r))^{-\frac{1}{4}}| &\lesssim 1, \quad |\partial_\sigma[(-Q(\sigma^2 r))^{-\frac{1}{4}}]| \lesssim \sigma^{-1} \lesssim s^{\frac{1}{2}}, \quad |\partial_\sigma^2[(-Q(\sigma^2 r))^{-\frac{1}{4}}]| \lesssim s \\ O((\zeta_r(\sigma))^{-1}) &\lesssim 1, \quad |\partial_\sigma[(\zeta_r(\sigma))^{-1}]| \lesssim 1, \quad |\partial_\sigma^2[(\zeta_r(\sigma))^{-1}]| \lesssim \sigma^{-1} \lesssim s^{\frac{1}{2}} \\ |1 + \sigma b_\pm(\sigma, \zeta(\sigma^2 r))| &\lesssim 1, \quad \left|\frac{d}{d\sigma}[\sigma b_\pm(\sigma, \zeta(\sigma^2 r))]\right| \lesssim 1, \quad \left|\frac{d^2}{d\sigma^2}[\sigma b_\pm(\sigma, \zeta(\sigma^2 r))]\right| \lesssim s \end{aligned}$$

and then by the Leibniz rule

$$|\partial_\sigma a_\pm(\sigma, r)| \lesssim \sigma^{-1} \lesssim s^{\frac{1}{2}}, \quad |\partial_\sigma^2 a_\pm(\sigma, r)| \lesssim s$$

as claimed. \square

Remark 2.15. We remark that our connection in Corollary 2.14 is consistent with the asymptotic behavior of $M_{\frac{i}{2\sigma}, \frac{1}{2}}(2i\sigma r)$. Using [8, (13.14.32) and (13.14.21)] one can calculate the following asymptotic behavior as $\sigma r \rightarrow \infty$

$$(2.59) \quad M_{\frac{i}{2\sigma}, \frac{1}{2}}(2i\sigma r) \sim \frac{ie^{\frac{\pi}{4\sigma}}}{|\Gamma(1 + \frac{i}{2\sigma})|} \sin\left(\sigma r - \frac{1}{2\sigma} \log(2\sigma r) + \theta(\sigma)\right),$$

where $\theta(\sigma) := \arg(\Gamma(1 + i/(2\sigma)))$. Therefore, by (3.1) we have $e(\sigma, r) \sim \sin(\sigma r - \frac{1}{2\sigma} \log(2\sigma r) + \theta(\sigma))$ as $\sigma r \rightarrow \infty$. Here, we used the fact that $|\Gamma(is)| = \sqrt{\frac{\pi}{s \sinh(\pi s)}}$, see (5.4.3) in [8].

On the other hand, by Stirling's formula [8, (5.11.1)], we have as $\sigma \rightarrow 0$,

$$\theta(\sigma) = \frac{-\ln(2\sigma)}{2\sigma} - \frac{1}{2\sigma} + \frac{\pi}{4} - \frac{\sigma}{3} + O_2(\sigma^3)$$

and therefore,

$$e^{\pm i(\sigma r - \frac{1}{2\sigma} \log(2\sigma r) + \theta(\sigma))} = e^{\pm i(\sigma r - \frac{1}{2\sigma} - \frac{\ln(4\sigma^2 r)}{2\sigma} + \frac{\pi}{4})} (1 + O(\sigma)).$$

By [8, (10.4.60), (10.4.64)], when $x > 0$ is big

$$\begin{aligned}\text{Ai}(-x) &= \frac{1}{\sqrt{\pi}x^{1/4}} \sin\left(\frac{2}{3}x^{3/2} + \frac{\pi}{4}\right) + O\left(\frac{1}{x^{3/2}}\right), \\ \text{Bi}(-x) &= \frac{1}{\sqrt{\pi}x^{1/4}} \cos\left(\frac{2}{3}x^{3/2} + \frac{\pi}{4}\right) + O\left(\frac{1}{x^{3/2}}\right).\end{aligned}$$

Then the consistency is now clear using the following expansions

$$\begin{aligned}q^{-\frac{1}{4}}\psi_+(\tau(\sigma^2 r)) &= \frac{\sigma^{\frac{1}{6}}}{\pi^{\frac{1}{2}}} e^{-i(\sigma r - \frac{1}{2\sigma} - \frac{\ln(4\sigma^2 r)}{2\sigma} - \frac{\pi}{4})} (1 + O((\sigma r)^{-1})) \\ q^{-\frac{1}{4}}\psi_-(\tau(\sigma^2 r)) &= \frac{\sigma^{\frac{1}{6}}}{\pi^{\frac{1}{2}}} e^{i(\sigma r - \frac{1}{2\sigma} - \frac{\ln(4\sigma^2 r)}{2\sigma} - \frac{\pi}{4})} (1 + O((\sigma r)^{-1}))\end{aligned}$$

as $(\sigma r) \rightarrow \infty$ in Corollary 2.14.

We finish this section with the following lemma and its corollary.

Lemma 2.16. *Let c be sufficiently small such that for all $\sigma < c$ (2.39) and (2.58) hold. Moreover let $\frac{3}{4} \leq n < 1$, $m < \infty$ and define $\chi_n^m := \tilde{\chi}_n \chi_m$. Then one has*

$$\begin{aligned}|\chi_c(\sigma)\chi_n^m(\sigma^2 r)e(\sigma, r)| &\lesssim \chi_c(\sigma)\chi_n^m(\sigma^2 r)\sigma^2 r \\ |\chi_c(\sigma)\chi_n^m(\sigma^2 r)\partial_\sigma\{e(\sigma, r)\}| &\lesssim \chi_c(\sigma)\chi_n^m r \\ |\chi_c(\sigma)\chi_n^m(\sigma^2 r)\partial_\sigma^2\{e(\sigma, r)\}| &\lesssim \chi_c(\sigma)\chi_n^m(\sigma^2 r)\sigma^{-2} r.\end{aligned}$$

Proof. Recall by (2.39), and (2.58), we have

$$e(\sigma, r) = c_+(\sigma) \frac{\text{Ai}(-\sigma^{-\frac{2}{3}}\zeta(\sigma^2 r))}{q^{\frac{1}{4}}(\sigma^2 r)} (1 + \sigma a(\sigma, \zeta(\sigma^2 r))) + c_-(\sigma) \frac{\text{Bi}(-\sigma^{-\frac{2}{3}}\zeta(\sigma^2 r))}{q^{\frac{1}{4}}(\sigma^2 r)} (1 + \sigma b(\sigma, \zeta(\sigma^2 r)))$$

where a, b hold the bounds in Proposition 2.7 for $\sigma^2 r \leq 1$, and the bounds in Proposition 2.11 for $\sigma^2 r \geq 1$.

We show the statement first for the leading terms. We note that by the definition of χ_c , given $n \geq \frac{3}{4}$, the cut-off $\tilde{\chi}_n \chi_m(x)$ is supported for $x \geq \frac{1}{2}$. Therefore, by expansions (2.24), (2.44) and the fact that $-\frac{3}{2}(\zeta(\frac{1}{2}))^{\frac{2}{3}} = \frac{\pi}{4} - \frac{1}{2}$ we have

$$\begin{aligned}(2.60) \quad |\chi_c(\sigma)\chi_n^m(x) \text{Bi}(-\sigma^{-\frac{2}{3}}\zeta(x))| &\lesssim e^{\frac{1}{\sigma}(\frac{\pi}{4} - \frac{1}{2})} \sigma^{\frac{1}{6}} \\ |\chi_c(\sigma)\chi_n^m(x) \dot{\text{Bi}}(-\sigma^{-\frac{2}{3}}\zeta(x))| &\lesssim e^{\frac{1}{\sigma}(\frac{\pi}{4} - \frac{1}{2})} \sigma^{-\frac{5}{6}}, \\ |\chi_c(\sigma)\chi_n^m(x) \ddot{\text{Bi}}(-\sigma^{-\frac{2}{3}}\zeta(x))| &\lesssim e^{\frac{1}{\sigma}(\frac{\pi}{4} - \frac{1}{2})} \sigma^{-\frac{11}{6}},\end{aligned}$$

As usual we use $\dot{\cdot}$ for ζ -derivatives. Similarly

$$\begin{aligned}(2.61) \quad |\chi_c(\sigma)\chi_n^m(x) \text{Ai}(-\sigma^{-\frac{2}{3}}\zeta(x))| &\lesssim \sigma^{\frac{1}{6}}. \\ |\chi_c(\sigma)\chi_n^m(x) \dot{\text{Ai}}(-\sigma^{-\frac{2}{3}}\zeta(x))| &\lesssim \sigma^{-\frac{5}{6}} \\ |\chi_c(\sigma)\chi_n^m(x) \ddot{\text{Ai}}(-\sigma^{-\frac{2}{3}}\zeta(x))| &\lesssim \sigma^{-\frac{11}{6}}.\end{aligned}$$

Furthermore, by definition of q , we have for $x \geq \frac{1}{2}$

$$(2.62) \quad \partial_x^j \{\zeta(x)\} = \langle x \rangle^{2/3-j}, \quad \partial_x^j \{q^{-\frac{1}{4}}(x)\} = \langle x \rangle^{1/6-j}.$$

As a result, using (2.60), (2.62), and $c_-(\sigma)$ from Corollary 2.14 we obtain

$$(2.63) \quad \left| \chi_c(\sigma) \chi_n^m(\sigma^2 r) c_-(\sigma) \frac{\text{Bi}(-\sigma^{-\frac{2}{3}} \zeta(\sigma^2 r))}{q^{\frac{1}{4}}(\sigma^2 r)} \right| \lesssim \chi_n^m(\sigma^2 r) e^{-\frac{1}{\sigma}(\frac{\pi}{4}-\frac{1}{2})} \langle \sigma^2 r \rangle^{\frac{1}{6}} \lesssim e^{-\frac{1}{\sigma}(\frac{\pi}{4}-\frac{1}{2})} \sigma^2 r.$$

In the last inequality we used the fact that $n \leq \sigma^2 r$.

As usual to shorten the notation, for the rest of the proof we avoid using the variable $\sigma^2 r$ in ζ . We continue with estimating the σ derivative of the leading term. We compute

$$(2.64) \quad \begin{aligned} |\chi_c \chi_n^m \partial_\sigma \{c_-(\sigma) q^{-\frac{1}{4}}(\zeta) \text{Bi}(-\sigma^{-\frac{2}{3}} \zeta)\}| &\lesssim \chi_c \chi_n^m [c'_-(\sigma) (q(\sigma^2 r))^{-\frac{1}{4}} \text{Bi}(-\sigma^{-\frac{2}{3}} \zeta) \\ &\quad + c_-(\sigma) \frac{\partial(q(\sigma^2 r))^{-\frac{1}{4}}}{\partial \sigma} \text{Bi}(-\sigma^{-\frac{2}{3}} \zeta) + c_-(\sigma) (q(\sigma^2 r))^{-\frac{1}{4}} \text{Bi}(-\sigma^{-\frac{2}{3}} \zeta) \frac{d\zeta}{d\sigma}] \end{aligned}$$

Using (2.60), (2.62) and $|\frac{d\zeta(\sigma^2 r)}{d\sigma}| \lesssim \sigma r$, we estimate $|(2.64)| \lesssim e^{-\frac{1}{\sigma}(\frac{\pi}{4}-\frac{1}{2})} r$

Similarly, using (2.60), (2.62) and $|\frac{d\zeta(\sigma^2 r)}{d\sigma}| \lesssim \sigma r$, one can compute

$$(2.65) \quad |\chi_c \chi_n^m \partial_\sigma^2 \{c_-(\sigma) q^{-\frac{1}{4}}(\zeta) \text{Bi}(-\sigma^{-\frac{2}{3}} \zeta)\}| \lesssim e^{-\frac{1}{\sigma}(\frac{\pi}{4}-\frac{1}{2})} \sigma^{-2} r.$$

As expected from the computation of (2.64), the restricting term $\sigma^{-2} r$ in (2.65) arises when both of the two derivatives fall on $\text{Bi}(-\sigma^{-\frac{2}{3}} \zeta)$. This situation leads us to the bound $\sigma^{-2}(\sigma r)^2$. However, due to the constraint $r \leq m\sigma^{-2}$, we can simplify this estimate to $\sigma^{-2} r$. With a similar observation and using (2.61), (2.62) we have

$$(2.66) \quad \begin{aligned} |\chi_c \chi_n^m c_+(\sigma) q^{-\frac{1}{4}}(\zeta) \text{Ai}(-\sigma^{\frac{3}{2}} \zeta)| &\lesssim \sigma^2 r \\ |\chi_c \chi_n^m \partial_\sigma^2 \{c_+(\sigma) q^{-\frac{1}{4}}(\zeta) \text{Ai}(-\sigma^{\frac{3}{2}} \zeta)\}| &\lesssim \sigma^{-j} r, \quad j = 1, 2. \end{aligned}$$

Hence, we obtain the bounds for the leading terms.

We next estimate the error terms. We first start with $\sigma^2 r \leq 1$. For that we use Proposition 2.7. We will only estimate a . The bounds on b can be estimated similarly. We first note that $|a_1| \lesssim 1$, and

$$(2.67) \quad |\chi_c \chi_n^m \partial_\sigma \{a(\sigma, \zeta(\sigma^2 r))\}| \lesssim \chi_c \chi_n^m [|\partial_\sigma \{a(\sigma, \zeta(x))\}|_{x=\sigma^2 r} + |\dot{a}(\sigma, \zeta) \frac{d\zeta}{d\sigma}|] \lesssim \sigma^{-\frac{4}{3}}.$$

Furthermore,

$$(2.68) \quad \begin{aligned} |\chi_c \chi_n^m \partial_\sigma^2 \{a_1(\sigma, \zeta(\sigma^2 r))\}| &\lesssim \chi_c \chi_n^m [|\partial_\sigma^2 \{a_1(\sigma, \zeta(x))\}|_{x=\sigma^2 r} + |\partial_\sigma \{\dot{a}_1(\sigma, \zeta)\} \frac{d\zeta}{d\sigma}| \\ &\quad + |\ddot{a}_1(\sigma, \zeta) (\frac{d\zeta}{d\sigma})^2| + |\dot{a}_1(\sigma, \zeta) \frac{d^2 \zeta}{d\sigma^2}|] \lesssim \chi_n^m \sigma^{-\frac{4}{3}} r \lesssim \sigma^{-\frac{10}{3}}. \end{aligned}$$

We next, comment on the error term when $\sigma^2 r \geq 1$. For that we use Proposition 2.11 and will estimate σb_\pm . In fact, using σb_\pm in (2.67) and (2.68) instead of a_1 , we see the same bounds holds for the sigma derivatives of σb_\pm in the support of $\chi_c \chi_n^m$. Combining these bounds with

(2.63), (2.64), (2.65), (2.66) we obtain the statement. Here for the discontinuity at $\sigma^2 r = 1$, we note that originally we converge to $M_{\frac{i}{2\sigma}, \frac{1}{2}}(2i\sigma r)$ and this function is analytic in the vicinity of turning point. \square

The following corollary is due to Lemma 2.4 and Lemma 2.16.

Corollary 2.17. *Fix $c > 0$ sufficiently small and $k < \infty$. Then for any $\beta \geq 0$ we have*

$$\begin{aligned} |\chi_c(\sigma)\chi_k(\sigma^2 r)e(\sigma, r)| &\lesssim \chi_c(\sigma)\chi_k(\sigma^2 r)r\sigma^2, \\ |\chi_c(\sigma)\chi_k(\sigma^2 r)\partial_\sigma\{e(\sigma, r)\}| &\lesssim \chi_c(\sigma)r[\sigma^\beta\chi_{\frac{1}{2}}(\sigma^2 r) + \chi_{\frac{1}{2}}^k(\sigma^2 r)], \\ |\chi_c(\sigma)\chi_k(\sigma^2 r)\partial_\sigma^2\{e(\sigma, r)\}| &\lesssim \chi_c(\sigma)r[\sigma^\beta\chi_{\frac{1}{2}}(\sigma^2 r) + \sigma^{-2}\chi_{\frac{1}{2}}^k(\sigma^2 r)]. \end{aligned}$$

3. EIGENFUNCTION APPROXIMATION: LARGE ENERGIES

In this section, we will consider the energies when $\sigma \geq c > 0$ where $c \ll 1$. Recall (3.1), we have

$$\begin{aligned} (3.1) \quad e(\sigma, r) &= -i\sigma^{-\frac{1}{2}}[e^{\frac{\pi}{\sigma}} - 1]^{-\frac{1}{2}}M_{\frac{i}{2\sigma}, \frac{1}{2}}(2i\sigma r) \\ &= -i\sigma^{-\frac{1}{2}}[e^{\frac{\pi}{\sigma}} - 1]^{-\frac{1}{2}}e^{-i\sigma r}(2i\sigma r)M\left(1 - i/(2\sigma), 2, 2i\sigma r\right). \end{aligned}$$

Here $M(a, b, z)$ is the Kummer's function of the first kind. In this section we prove Proposition 3.1.

Proposition 3.1. *Let $k < \infty$, then the following expansions are valid for $e(\sigma, r)$*

$$(3.2) \quad \tilde{\chi}_c(\sigma)\chi_k(\sigma r)e(\sigma, r) = \tilde{\chi}_c(\sigma)\chi_k(\sigma r)\sigma^{\frac{1}{2}}[e^{\frac{\pi}{\sigma}} - 1]^{-\frac{1}{2}}r(1 + O_2(\sigma r))$$

$$(3.3) \quad \tilde{\chi}_c(\sigma)\tilde{\chi}_k(\sigma r)e(\sigma, r) = -\frac{i}{\sqrt{\pi}}\tilde{\chi}_c(\sigma)\tilde{\chi}_k(\sigma r)[e^{i(\sigma r - \frac{\ln(2\sigma r)}{\sigma} - \theta(\sigma))} + e^{-i(\sigma r - \frac{\ln(2\sigma r)}{\sigma} + \theta(\sigma))}] + \mathcal{E}(\sigma, r)$$

where $\theta(\sigma) = \arg(\Gamma(1 + i/(2\sigma)))$ and

$$(3.4) \quad |\mathcal{E}(\sigma, r)| \lesssim 1, \quad |\partial_\sigma\{\mathcal{E}(\sigma, r)\}| \lesssim r \quad |\partial_\sigma^2\{\mathcal{E}''(\sigma, r)\}| \lesssim \sigma^{-1}r.$$

Before we start the proof, we state a couple of expressions for $M(a, b, z)$. These formulas can be found in [8, Chapter 13]. We will use (3.5) to prove (3.2) and (3.6) to prove the (3.3). One has

$$(3.5) \quad M(a, b, z) = 1 + \sum_{s=1}^{\infty} \frac{a(a+1)\dots(a+(s-1))}{b(b+1)\dots(b+(s-1))s!} z^s$$

for any nonpositive integer b and

$$(3.6) \quad M(a, b, z) = \frac{1}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zs} s^{a-1} (1-s)^{b-a-1} dt.$$

for $\Re(b) > \Re(a) > 0$.

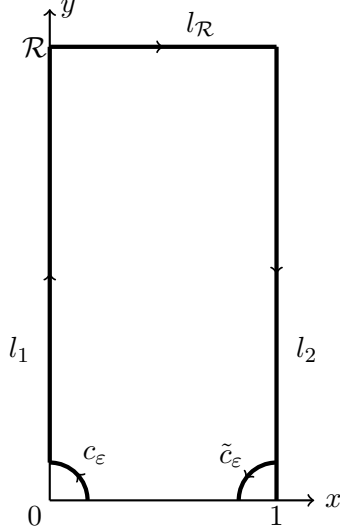
We start with the following lemma which analyzes the integral in (3.6) for $a = 1 - \frac{i}{2\sigma}$, $b = 2$ and $z = 2i\sigma r$.

Lemma 3.2. *We have the following expansion*

$$(3.7) \quad \begin{aligned} \tilde{\chi}_c(\sigma)\tilde{\chi}_c(\sigma r) \int_0^1 e^{2i\sigma r s} s^{-\frac{i}{2\sigma}} (1-s)^{\frac{i}{2\sigma}} ds \\ = \frac{\tilde{\chi}_c(\sigma)\tilde{\chi}_c(\sigma r)}{(2i\sigma r)} \left[e^{2i\sigma r} (2\sigma r i)^{-\frac{i}{2\sigma}} \Gamma\left(1 + \frac{i}{2\sigma}\right) - (-2i\sigma r)^{\frac{i}{2\sigma}} \Gamma\left(1 - \frac{i}{2\sigma}\right) \right] \\ + e^{\frac{\pi}{2\sigma}} \left[b_+(\sigma, r) + e^{2i\sigma r} b_-(\sigma, r) \right] \end{aligned}$$

where $|b_{\pm}(\sigma, r)| \lesssim |\sigma r|^{-2}$, $|\partial_{\sigma}^j \{b_{\pm}(\sigma, r)\}| \lesssim \sigma^{-j} |\sigma r|^{-1}$ for $j = 0, 1, 2$.

Proof. Using the contour below, we obtain (3.8).



$$(3.8) \quad \begin{aligned} \int_{\varepsilon}^{1-\varepsilon} e^{2i\sigma r s} s^{-\frac{i}{2\sigma}} (1-s)^{\frac{i}{2\sigma}} ds &= \int_{c_{\varepsilon}} e^{2i\sigma r s} s^{-\frac{i}{2\sigma}} (1-s)^{\frac{i}{2\sigma}} ds \\ &+ \int_{l_1} e^{2i\sigma r s} s^{-\frac{i}{2\sigma}} (1-s)^{\frac{i}{2\sigma}} ds + \int_{l_{\mathcal{R}}} e^{2i\sigma r s} s^{-\frac{i}{2\sigma}} (1-s)^{\frac{i}{2\sigma}} ds \\ &+ \int_{l_2} e^{2i\sigma r s} s^{-\frac{i}{2\sigma}} (1-s)^{\frac{i}{2\sigma}} ds + \int_{\tilde{c}_{\varepsilon}} e^{2i\sigma r s} s^{-\frac{i}{2\sigma}} (1-s)^{\frac{i}{2\sigma}} ds. \end{aligned}$$

Note that if $\sigma \geq c > 0$, then

$$\begin{aligned} \left| \int_{c_{\varepsilon}} e^{2i\sigma r s} s^{-\frac{i}{2\sigma}} (1-s)^{\frac{i}{2\sigma}} ds \right| &= \varepsilon \left| \int_{\pi/2}^{\pi} e^{2i\sigma r \varepsilon e^{is}} (\varepsilon e^{is})^{-\frac{i}{2\sigma}} (1 - \varepsilon e^{is})^{\frac{i}{2\sigma}} e^{is} ds \right| \leq \varepsilon e^{\frac{\pi}{2c}} \\ \left| \int_{\tilde{c}_{\varepsilon}} e^{2i\sigma r s} s^{-\frac{i}{2\sigma}} (1-s)^{\frac{i}{2\sigma}} ds \right| &= \varepsilon \left| \int_{\pi/2}^{\pi} e^{2i\sigma r (1 + \varepsilon e^{is})} (1 + \varepsilon e^{is})^{-\frac{i}{2\sigma}} (-\varepsilon e^{is})^{\frac{i}{2\sigma}} e^{is} ds \right| \leq \varepsilon e^{\frac{\pi}{2c}}. \end{aligned}$$

Hence, the first and last term on the right side of the equality in (3.8) goes to zero as $\varepsilon \rightarrow 0$.

Moreover, as $\sigma r \geq c$

$$\left| \int_{l_{\mathcal{R}}} e^{2i\sigma r s} s^{-\frac{i}{2\sigma}} (1-s)^{\frac{i}{2\sigma}} ds \right| = \left| \int_0^1 e^{2i\sigma r (i\mathcal{R} + s)} (i\mathcal{R} + s)^{-\frac{i}{2\sigma}} (1 - (i\mathcal{R} + s))^{\frac{i}{2\sigma}} ds \right| \leq e^{-2\mathcal{R}c} e^{\frac{\pi}{2c}}.$$

Therefore, the third term on the right side of the equality in (3.8) goes to zero as $\mathcal{R} \rightarrow \infty$.

We next estimate the second and fourth terms on the right side of the equality in (3.8). Parametrizing the paths and letting $\varepsilon \rightarrow 0$, $\mathcal{R} \rightarrow \infty$ we have

$$(3.9) \quad \tilde{\chi}_c(\sigma)\tilde{\chi}_c(\sigma r) \int_0^1 e^{2i\sigma r s} s^{-\frac{i}{2\sigma}} (1-s)^{\frac{i}{2\sigma}} ds = \tilde{\chi}_c(\sigma)\tilde{\chi}_c(\sigma r)(A_1 + A_2)$$

where

$$A_1(\sigma, r) := \int_0^\infty e^{2i\sigma r(is)} (is)^{-\frac{i}{2\sigma}} (1-is)^{\frac{i}{2\sigma}} i ds$$

$$A_2(\sigma, r) := \int_0^\infty e^{2i\sigma r(1+is)} (1+is)^{-\frac{i}{2\sigma}} (-is)^{\frac{i}{2\sigma}} (-i) ds.$$

We can write

$$(3.10) \quad A_1(\sigma, r) = \int_0^\infty e^{-2\sigma r s} (is)^{-\frac{i}{2\sigma}} i ds + \int_0^\infty e^{-2\sigma r s} (is)^{-\frac{i}{2\sigma}} [(1-is)^{\frac{i}{2\sigma}} - 1] i ds$$

$$= (-2i\sigma r)^{-1 + \frac{i}{2\sigma}} \Gamma(1 - i/(2\sigma)) + e^{\frac{\pi}{2\sigma}} \int_0^\infty e^{-\frac{\pi}{2\sigma}} e^{-2\sigma r s} (is)^{-\frac{i}{2\sigma}} [(1-is)^{\frac{i}{2\sigma}} - 1] i ds$$

and similarly

$$A_2(\sigma, r) = e^{2i\sigma r} \int_0^\infty e^{-2\sigma r s} (1+is)^{-\frac{i}{2\sigma}} (-is)^{\frac{i}{2\sigma}} (-i) ds$$

$$= e^{2i\sigma r} \int_0^\infty e^{-2\sigma r s} (-is)^{\frac{i}{2\sigma}} (-i) ds - e^{2i\sigma r} \int_0^\infty e^{-2\sigma r s} [(1+is)^{-\frac{i}{2\sigma}} - 1] (-is)^{\frac{i}{2\sigma}} i ds$$

$$= e^{2i\sigma r} (2\sigma r i)^{-1 - \frac{i}{2\sigma}} \Gamma(1 + i/(2\sigma)) - e^{2i\sigma r} \int_0^\infty e^{-2\sigma r s} [(1+is)^{-\frac{i}{2\sigma}} - 1] (-is)^{\frac{i}{2\sigma}} i ds.$$

If we let

$$b_+(\sigma, r) := \int_0^\infty e^{-\frac{\pi}{2\sigma}} e^{-2\sigma r s} (is)^{-\frac{i}{2\sigma}} [(1-is)^{\frac{i}{2\sigma}} - 1] i ds$$

$$b_-(\sigma, r) := - \int_0^\infty e^{-\frac{\pi}{2\sigma}} e^{-2\sigma r s} [(1+is)^{-\frac{i}{2\sigma}} - 1] (-is)^{\frac{i}{2\sigma}} i ds$$

then plugging A_1, A_2 in (3.9) and comparing to (3.7), one can see that it is enough to show that b_k satisfies the bounds $|\partial_\sigma^j \{b_\pm(\sigma, r)\}| \lesssim \sigma^{-j} |\sigma r|^{-1}$, $|b_\pm(\sigma, r)| \lesssim |\sigma r|^{-2}$. Below, we prove the bounds for b_+ , the bounds for b_- follow similarly.

Observe that $|(is)^{-\frac{i}{2\sigma}}[(1-is)^{\frac{i}{2\sigma}}-1]| \lesssim 1$. Furthermore, for any $s < 1$, $(1-is)^{\frac{i}{2\sigma}} = 1 + O(s/\sigma)$ and therefore

$$|(is)^{-\frac{i}{2\sigma}}[(1-is)^{\frac{i}{2\sigma}}-1]| \lesssim \sigma^{-1}s.$$

This allows us to deduce, through interpolation, the following inequality for any $\sigma \geq c$:

$$(3.11) \quad |b_+(\sigma, r)| \lesssim \sigma^{-\alpha} \int_0^{1/|\sigma r|} s^\alpha ds + \int_{1/|\sigma r|}^\infty \frac{1}{(\sigma r s)^{5/2}} ds \lesssim |\sigma r|^{-(1+\alpha)}.$$

We next estimate $\partial_\sigma b_1(\sigma, r)$. Note that since b_1 is convergent we can differentiate under the integral sign. Hence, we first estimate the σ derivative of the integrand in $b_1(\sigma, r)$. One has

$$(3.12) \quad \begin{aligned} \partial_\sigma \{e^{-\frac{\pi}{2\sigma}} e^{-2\sigma r s} (is)^{-\frac{i}{2\sigma}} [(1-is)^{\frac{i}{2\sigma}} - 1]\} \\ = (is)^{-\frac{i}{2\sigma}} \partial_\sigma \{e^{-\frac{\pi}{2\sigma}} e^{-2\sigma r s} [(1-is)^{\frac{i}{2\sigma}} - 1]\} \\ + e^{-\frac{\pi}{2\sigma}} e^{-2\sigma r s} [(1-is)^{\frac{i}{2\sigma}} - 1] \partial_\sigma \{(is)^{-\frac{i}{2\sigma}}\}. \end{aligned}$$

Using the fact that $e^{-2\sigma r s} \lesssim \langle \sigma r s \rangle^{-\ell}$ for any $\ell > 0$, the first term in (3.12) can be estimated as

$$(3.13) \quad \begin{aligned} |(is)^{-\frac{i}{2\sigma}} \partial_\sigma \{e^{-\frac{\pi}{2\sigma}} e^{-2\sigma r s} [(1-is)^{\frac{i}{2\sigma}} - 1]\}| &\lesssim e^{-2\sigma r s} [(rs) + \sigma^{-2} \langle s \rangle^{0+}] \\ &\lesssim \langle \sigma r s \rangle^{-5/2} [(rs) + \sigma^{-2} \langle s \rangle^{0+}]. \end{aligned}$$

For the second term, we first have

$$(3.14) \quad \tilde{\chi}_c(\sigma) |e^{-\frac{\pi}{4\sigma}} [(1-is)^{\frac{i}{2\sigma}} - 1]| \lesssim \frac{s}{\sigma} \chi(s < 1) + \chi(s \geq 1).$$

Therefore, the second term in (3.12) can be estimated for $\sigma \geq c$ as

$$(3.15) \quad |e^{-\frac{\pi}{2\sigma}} e^{-2\sigma r s} [(1-is)^{\frac{i}{2\sigma}} - 1] \partial_\sigma \{(is)^{-\frac{i}{2\sigma}}\}| \lesssim \langle \sigma r s \rangle^{-3/2} \left[\frac{s}{\sigma} \chi(s < 1) + \chi(s \geq 1) \right] \frac{\log |s|}{\sigma^2}.$$

Using (3.13), (3.15) and (3.12) we obtain

$$(3.16) \quad \begin{aligned} |\tilde{\chi}_c(\sigma) \tilde{\chi}_c(\sigma r) \partial_\sigma b_+(\sigma, r)| &\lesssim \tilde{\chi}_c(\sigma) \tilde{\chi}_c(\sigma r) \int_0^{1/|\sigma r|} [\sigma^{-3} \max\{s^{1-}, s^{0+}\} + \sigma^{-1}] ds \\ &+ \tilde{\chi}_c(\sigma) \tilde{\chi}_c(\sigma r) \int_{|\sigma r|^{-1}}^\infty [(rs)(\sigma r s)^{-5/2} + \sigma^{-2} s^{0+} (\sigma r s)^{-3/2}] ds \lesssim \sigma^{-1} |\sigma r|^{-1}. \end{aligned}$$

We next estimate the second derivative of the integrand in $b_1(\sigma, r)$. One has,

$$(3.17) \quad \begin{aligned} \partial_\sigma^2 \{e^{-\frac{\pi}{2\sigma}} e^{-2\sigma r s} (is)^{-\frac{i}{2\sigma}} [(1-is)^{\frac{i}{2\sigma}} - 1]\} &= (is)^{-\frac{i}{2\sigma}} \partial_\sigma^2 \{e^{-\frac{\pi}{2\sigma}} e^{-2\sigma r s} [(1-is)^{\frac{i}{2\sigma}} - 1]\} \\ &+ e^{-\frac{\pi}{2\sigma}} e^{-2\sigma r s} [(1-is)^{\frac{i}{2\sigma}} - 1] \partial_\sigma^2 \{(is)^{-\frac{i}{2\sigma}}\} \\ &+ 2\partial_\sigma \{e^{-\frac{\pi}{4\sigma}} e^{-2\sigma r s} [(1-is)^{\frac{i}{2\sigma}} - 1]\} \partial_\sigma \{e^{-\frac{\pi}{4\sigma}} (is)^{-\frac{i}{2\sigma}}\}. \end{aligned}$$

We have the following estimate for the first term in (3.17).

$$(3.18) \quad \tilde{\chi}_c(\sigma) |(is)^{-\frac{i}{2\sigma}} \partial_\sigma^2 \{e^{-\frac{\pi}{2\sigma}} e^{-2\sigma rs} [(1-is)^{\frac{i}{2\sigma}} - 1]\}| \\ \lesssim \langle \sigma rs \rangle^{-7/2} [(rs)^2 + (rs) \langle s \rangle^{0+} \sigma^{-2} + \sigma^{-3} \langle s \rangle^{0+}].$$

Moreover, using (3.14) we have

$$(3.19) \quad \tilde{\chi}_c(\sigma) |e^{-\frac{\pi}{2\sigma}} e^{-2\sigma rs} [(1-is)^{\frac{i}{2\sigma}} - 1] \partial_\sigma^2 \{(is)^{-\frac{i}{2\sigma}}\}| \\ \lesssim \langle \sigma rs \rangle^{-3/2} [\frac{S}{\sigma} \chi(s < 1) + \chi(s \geq 1)] \sigma^{-3} \max\{s^{0+}, s^{0-}\}.$$

and

$$(3.20) \quad \tilde{\chi}_c(\sigma) \partial_\sigma \{e^{-\frac{\pi}{4\sigma}} e^{-2\sigma rs} [(1-is)^{\frac{i}{2\sigma}} - 1]\} \\ \lesssim \langle \sigma rs \rangle^{-3/2} [(rs) + \sigma^{-2} s \chi(s < 1) + \sigma^{-2} s^{0+} \chi(s \geq 1)].$$

Using (3.20) we also estimate the last term in (3.17) as

$$(3.21) \quad \tilde{\chi}_c(\sigma) |\partial_\sigma \{e^{-\frac{\pi}{4\sigma}} e^{-2\sigma rs} [(1-is)^{\frac{i}{2\sigma}} - 1]\} \partial_\sigma \{e^{-\frac{\pi}{4\sigma}} (is)^{-\frac{i}{2\sigma}}\}| \\ \lesssim \langle \sigma rs \rangle^{-3/2} [(rs) + \sigma^{-2} s \chi(s < 1) + \sigma^{-2} s^{0+} \chi(s \geq 1)] \sigma^{-2} \log |s|.$$

Using (3.18), (3.20), (3.21) and (3.17) in a similar way to (3.16), we obtain

$$(3.22) \quad |\tilde{\chi}_c(\sigma) \tilde{\chi}_c(\sigma r) \partial_\sigma^2 b_+(\sigma, r)| \lesssim \sigma^{-2} |\sigma r|^{-1}.$$

The estimates (3.11), (3.16) and (3.22) establishes the statement for $b_1(\sigma, r)$. \square

Proof of Proposition 3.1. We first prove (3.2). Note that by (3.5) we have

$$(3.23) \quad M\left(1 - i/(2\sigma), 2, 2i\sigma r\right) = 1 + i\sigma r + \frac{r}{2} + E(\sigma, r)$$

where

$$E(\sigma, r) = \left(1 - \frac{i}{2\sigma}\right) \left(2 - \frac{i}{2\sigma}\right) (\sigma r)^2 \sum_{s=0}^{\infty} c_s(\sigma) (\sigma r)^s.$$

Since $\limsup_{s \rightarrow \infty} \left| \frac{c_{s+1}(\sigma)}{c_s(\sigma)} \right| = 0$, the sum is convergent in the support of $\chi_k(\sigma r)$. Moreover, in the support of $\tilde{\chi}_c(\sigma)$ one has $r \lesssim (\sigma r)$ and therefore,

$$|\partial_\sigma^j \{E(\sigma, r)\}| \lesssim \sigma^{2-j} r^2.$$

Using (3.23), the expansion $e^{-i\sigma r} = 1 - i\sigma r + O_2((\sigma r)^2)$ for $\sigma r < 1$ and

$$(3.24) \quad \frac{e^{\frac{\pi}{4\sigma}}}{|\Gamma(1 + \frac{i}{2\sigma})|} = (\pi)^{-\frac{1}{2}} \sigma^{\frac{1}{2}} [e^{\frac{\pi}{\sigma}} - 1]^{\frac{1}{2}}$$

in (3.1), we obtain the statement.

For the proof of (3.3) we use Lemma 3.2. Note that by (3.6) we have

$$(3.25) \quad M(1 - i/(2\sigma), 2, 2i\sigma r) = \frac{1}{|\Gamma(1 + i/(2\sigma))|^2} \int_0^1 e^{2i\sigma r s} s^{-\frac{i}{2\sigma}} (1-s)^{\frac{i}{2\sigma}} ds.$$

Moreover,

$$\Gamma(1 \pm i/(2\sigma)) = |\Gamma(1 + i/(2\sigma))| e^{\pm i\theta(\sigma)}$$

where $\theta(\sigma) = \arg(\Gamma(1 + i/(2\sigma)))$ and

$$(\mp 2i\sigma r)^{\pm \frac{i}{2\sigma}} = e^{\frac{\pi}{4\sigma}} e^{\pm \frac{\log(2\sigma r)}{\sigma}}.$$

Therefore, using Lemma 3.2, (3.25) and (3.24) in (3.1), we have in the support of $\tilde{\chi}_c(\sigma)\tilde{\chi}_c(\sigma r)$

$$\begin{aligned} e(\sigma, \sigma^2 r) &= -i(\pi)^{-\frac{1}{2}} [e^{-i(\sigma r - \sigma^{-1} \ln(2\sigma r) + \theta(\sigma))} + e^{i(\sigma r - \sigma^{-1} \ln(2\sigma r) + \theta(\sigma))}] \\ &\quad - i(\pi)^{-\frac{1}{2}} (2i\sigma r) \frac{e^{\frac{\pi}{4\sigma}}}{|\Gamma(1 + \frac{i}{2\sigma})|} [e^{-i\sigma r} b_+(\sigma, r) + e^{i\sigma r} b_-(\sigma, r)]. \end{aligned}$$

Letting $\mathcal{E}(\sigma, r) := \pi^{-1}(2\sigma r)\sigma^{\frac{1}{2}} [e^{\frac{\pi}{\sigma}} - 1]^{\frac{1}{2}} [e^{-i\sigma r} b_+(\sigma, r) + e^{i\sigma r} b_-(\sigma, r)]$, we see that $\mathcal{E}(\sigma, r)$ holds the required bounds. We skip the validation of the bounds as it is basic differentiation. \square

4. PROOF OF THEOREM 1.1

In this section, we are focusing on estimating the kernel given by the equation:

$$K_t(r, s) = \frac{1}{2rs} \int_0^\infty e^{itq^2\sigma^2} e(q\sigma, r) e(q\sigma, s) d\sigma$$

as $\sup_{r,s} |K_t(r, s)| \lesssim t^{-\frac{3}{2}}$ for $r, s \geq 0$ and $t \geq 1$. Importantly, this bound is sufficient to establish the validity of Theorem 1.1 as one has

$$\|e^{itH_{0,q}} g\|_{L^\infty(\mathbb{R}^3)} = \left\| \int_0^\infty K_t(r, s) s^2 g(s) ds \right\|_{L^\infty(\mathbb{R}^3)} \lesssim \sup_{r,s} |K_t(r, s)| \|g\|_{L^1, s^2([0, \infty))}.$$

We normalize $q = 1$ and chose $\sigma < 1$ sufficiently small to write

$$\begin{aligned} K_t(r, s) &= \frac{1}{rs} \int_0^\infty e^{it\sigma^2} \chi_c(\sigma) e(\sigma, r) e(\sigma, s) d\sigma + \frac{1}{rs} \int_0^\infty e^{it\sigma^2} \tilde{\chi}_c(\sigma) e(\sigma, r) e(\sigma, s) d\sigma \\ &= K_t^l(r, s) + K_t^h(r, s). \end{aligned}$$

4.1. **Estimation of $K_t^l(r, s)$.** In this section, we will prove that

Proposition 4.1. *For any $c > 0$ sufficiently small, we have that*

$$\sup_{r,s} |K_t^l(r, s)| \lesssim t^{-\frac{3}{2}}.$$

We prove Proposition 4.1 with a series of lemmas. Fix some constant $k \geq 4$ and write

$$\begin{aligned} (4.1) \quad K_t^l(r, s) &= \frac{1}{rs} \int_0^\infty e^{it\sigma^2} \chi_c(\sigma) \chi_k(\sigma^2 r) \chi_k(\sigma^2 s) e(\sigma, r) e(\sigma, s) d\sigma \\ &\quad + \frac{1}{rs} \int_0^\infty e^{it\sigma^2} \chi_c(\sigma) [\chi_k(\sigma^2 r) \tilde{\chi}_k(\sigma^2 s) + \tilde{\chi}_k(\sigma^2 r) \chi_k(\sigma^2 s)] e(\sigma, r) e(\sigma, s) d\sigma \\ &\quad + \frac{1}{rs} \int_0^\infty e^{it\sigma^2} \chi_c(\sigma) \tilde{\chi}_k(\sigma^2 s) \tilde{\chi}_k(\sigma^2 r) e(\sigma, r) e(\sigma, s) d\sigma \\ &= K_1(r, s; t) + K_2(r, s; t) + K_3(r, s; t). \end{aligned}$$

By symmetry, we may always assume that $r \geq s$. Furthermore, observe that the support of $\chi_c(\sigma) \tilde{\chi}_k(\sigma^2 r)$ is empty unless $r \geq \frac{k}{c^2} > 1$ so we are free to assume that $r \geq s > 1$ when considering K_3 .

Lemma 4.2. *We have that $|K_1(r, s; t)| \lesssim t^{-\frac{3}{2}}$.*

Proof. Let $a(\sigma; s, r) = (rs)^{-1} \chi_c(\sigma) \chi_k(\sigma^2 r) \chi_k(\sigma^2 s) e(\sigma, r) e(\sigma, s)$. With $'$ denoting the derivative respect to σ , it is easy to see that, as a function of σ , $\chi_k(\sigma^2 s) = O_\infty(\sigma^0)$ from the computation $\chi_k'(\sigma^2 r) = \chi'(\sigma^2 r)(2\sigma r)$ and the fact that $\sigma^2 r \sim 1$ on the support of $\chi'(\sigma^2 r)$. Therefore, we may use the bounds from Corollary 2.17 to see that

$$(4.2) \quad |a(\sigma; r, s)| \lesssim \sigma^4, \quad |a'(\sigma; r, s)| \lesssim \sigma^2, \quad |a''(\sigma; r, s)| \lesssim 1.$$

Integrating by parts via the identity $e^{it\sigma^2} = (i2t\sigma)^{-1} \frac{d}{d\sigma} [e^{it\sigma^2}]$ and suppressing the variables r and s , we obtain

$$K_1(r, s; t) = \frac{1}{2it} \int_0^\infty e^{it\sigma^2} \left(\frac{a(\sigma)}{\sigma} \right)' d\sigma = \frac{1}{2it} \int_{\sigma < t^{-\frac{1}{2}}} e^{it\sigma^2} \left(\frac{a(\sigma)}{\sigma} \right)' d\sigma + \frac{1}{2it} \int_{\sigma \geq t^{-\frac{1}{2}}} e^{it\sigma^2} \left(\frac{a(\sigma)}{\sigma} \right)' d\sigma.$$

By (4.2), we have $\left| \left(\frac{a(\sigma, r, s)}{\sigma} \right)' \right| \lesssim \sigma r s$, therefore, the first term is bounded by $t^{-\frac{3}{2}} r s$. We apply another integration by parts to the second term to bound it by

$$(4.3) \quad t^{-2} \int_{\sigma \geq t^{-\frac{1}{2}}} \left| \frac{1}{\sigma} \left(\frac{a(\sigma)}{\sigma} \right)'' \right| + \left| \frac{1}{\sigma^2} \left(\frac{a(\sigma)}{\sigma} \right)' \right| d\sigma \lesssim \int_{\sigma \geq t^{-\frac{1}{2}}} \sigma^{-2} d\sigma \lesssim t^{-\frac{3}{2}}.$$

Here, we omit the boundary term arising from the integration by parts since it is bounded by the integral in (4.3). This finishes the proof. \square

To estimate the other terms in (4.1), we prove the following oscillatory integral estimate.

Lemma 4.3. *Suppose that for all $r > \frac{k}{c^2}$, $\omega_r(\sigma) : [0, \infty) \rightarrow \mathbb{R}$ is a C^2 phase function and $\delta_r : \mathbb{R} \rightarrow \mathbb{R}$ is a weight function satisfying*

- (1) $0 < \delta_r \lesssim \omega_r'(\sigma) \lesssim r$
- (2) $\omega_r''(\sigma) < 0$ and $|\omega_r''(\sigma)| \lesssim \frac{\delta_r}{\sigma}$

and $a_r(\sigma) : [0, \infty) \rightarrow \mathbb{C}$ is a C^2 amplitude function satisfying

- (1) $|a_r(\sigma)| \lesssim \frac{\sigma^2}{r}$
- (2) $|a_r'(\sigma)| \lesssim \sigma^2$
- (3) $\int_0^\infty \sigma^{-1} (|a_r''(\sigma)| + r|a_r'(\sigma)|) \chi(\sigma) d\sigma \lesssim 1$

uniformly for $\sigma \in [k^{\frac{1}{2}}r^{-\frac{1}{2}}, c]$ and $r > 0$. Then with $\chi(\sigma) := \chi_c(\sigma)\tilde{\chi}_k(\sigma^2r)$ we have that

$$I^\pm(r, s; t) := \int_0^\infty e^{i(t\sigma^2 \pm \omega_r(\sigma))} \chi(\sigma) a_r(\sigma) d\sigma \lesssim t^{-\frac{3}{2}}$$

with an implicit constant that does not depend on a_r or ω_r .

Remark 4.4. *In the application of this lemma, the phase and amplitude may depend additionally on the parameter s . The last sentence of the statement indicates that as long as the bounds on ω_r and a_r hold uniformly in s , the conclusion will also hold uniformly in s .*

Proof. As before, we first integrate by parts via $e^{it\sigma^2} = (2it\sigma)^{-1} \frac{d}{d\sigma} [e^{it\sigma^2}]$ to find that

$$\begin{aligned} I^\pm(r, s; t) &= \frac{1}{2it} \int_0^\infty e^{i(t\sigma^2 \pm \omega_r(\sigma))} [b_\pm(\sigma; r) + \tilde{b}(\sigma; r)] \chi(\sigma) d\sigma \\ &= I_1^\pm + I_2^\pm \end{aligned}$$

for

$$b_\pm(\sigma; r) = \pm i\sigma^{-1} \omega_r'(\sigma) a_r(\sigma) \quad \tilde{b}(\sigma; r) = \sigma^{-1} [\chi(\sigma) a_r(\sigma)]' - \sigma^{-2} a_r(\sigma).$$

We first estimate I_2^\pm . Split the integral as

$$I_2^\pm = \frac{1}{2it} \int_0^{t^{-\frac{1}{2}}} e^{i(t\sigma^2 \pm \omega_r(\sigma))} \tilde{b}(\sigma; r) d\sigma + \frac{1}{2it} \int_{t^{-\frac{1}{2}}}^\infty e^{i(t\sigma^2 \pm \omega_r(\sigma))} \tilde{b}(\sigma; r) d\sigma$$

and observe that the assumptions on a_r imply that

$$\begin{aligned} |\tilde{b}(\sigma; r)| &\lesssim \sigma^{-1} |a_r'(\sigma)| + \sigma^{-2} |a_r(\sigma)| \lesssim \sigma, \\ |\tilde{b}'(\sigma; r)| &\lesssim \sigma^{-1} |a_r''(\sigma)| + \sigma^{-2} |a_r'(\sigma)| + \sigma^{-3} |a_r(\sigma)| \lesssim \sigma^{-1} [|a_r''(\sigma)| + r|a_r'(\sigma)|] \end{aligned}$$

where for the final term in the second line we have used that $\sigma^{-3}|a_r(\sigma)| \lesssim 1/(\sigma r) \lesssim 1$. Therefore, the first integral is bounded by $t^{-\frac{3}{2}}$ and for the second we apply another integration by parts (ignoring the easily estimated boundary term) to bound it by

$$t^{-2} \int_{t^{-\frac{1}{2}}}^{\infty} \left(|[\sigma^{-1}\tilde{b}'(\sigma; r)]'| + |\sigma^{-1}\tilde{b}(\sigma; r)\omega_r'(\sigma)| \right) \chi(\sigma) d\sigma$$

$$\lesssim t^{-\frac{3}{2}} \int_0^{\infty} \sigma^{-1} (|a_r''(\sigma)| + r|a_r'(\sigma)|) \chi(\sigma) d\sigma \lesssim t^{-3/2}.$$

We now turn our attention to I_1^\pm . Here, we must treat the $\omega_r(\sigma)$ term as part of the phase so we write

$$I_1^\pm = (2it)^{-1} \int_0^\infty e^{it\Phi_{r,t}^\pm(\sigma)} b_\pm(\sigma; r) \chi(\sigma) d\sigma \quad \text{with} \quad \Phi_{r,t}(\sigma) := \sigma^2 \pm t^{-1}\omega_r(\sigma).$$

We have $(\Phi_{r,t}^\pm)'(\sigma) = 2\sigma \pm t^{-1}\omega_r'(\sigma)$, and $(\Phi_{r,t}^\pm)''(\sigma) = 2 \pm t^{-1}\omega_r''(\sigma)$. As $\omega_r' > 0$ and $\omega_r'' < 0$, only $\Phi_{r,t}^-$ has a stationary point and it is automatically non-degenerate.

Since the phase in I_1^+ is non-stationary, the integral is easily estimated. As before, the integrand is $O(\sigma)$ so we may split the domain of integration at $t^{-\frac{1}{2}}$ and integrate by parts to find that

$$(4.4) \quad I_1^+ \lesssim t^{-\frac{3}{2}} + t^{-2} \int_{t^{-\frac{1}{2}}}^{\infty} \left| \frac{b_+(\sigma; r)}{(\Phi_{r,t}^+(\sigma))'} + b_+(\sigma; r) \frac{d}{d\sigma} [((\Phi_{r,t}^+)')^{-1}(\sigma)] \right| \chi(\sigma) d\sigma.$$

Now by applying various properties of a_r and ω_r , we observe that

$$(4.5) \quad |b_\pm(\sigma; r)| \lesssim \sigma^{-1} r a_r(\sigma) \lesssim \sigma,$$

$$|b'_\pm(\sigma; r)| \lesssim \sigma^{-2} |[\omega_r' a_r](\sigma)| + \sigma^{-1} |[\omega_r'' a_r](\sigma)| + \sigma^{-1} |[\omega_r' a_r'](\sigma)| \lesssim 1 + \sigma^{-1} r |a_r'(\sigma)|$$

so that because $|(\Phi_{r,t}^+)''(\sigma)|^{-1} \lesssim \sigma^{-1}$ we have

$$\left| \frac{b'_+(\sigma; r)}{(\Phi_{r,t}^+(\sigma))'} \right| \lesssim \sigma^{-1} + \sigma^{-2} r |a_r'(\sigma)|$$

which makes an admissible contribution. Furthermore, observe that

$$\frac{|(\Phi_{r,t}^+)''(\sigma)|}{|(\Phi_{r,t}^+)''(\sigma)|^2} \lesssim \frac{2 + \delta_r/(t\sigma)}{(2\sigma + \delta_r/t)^2}$$

so that

$$\left| b_+(\sigma; r) \frac{d}{d\sigma} [(\Phi_{r,t}^+)^{-1}](\sigma) \right| \lesssim (2\sigma + \delta_r/t)^{-1} < \sigma^{-1}.$$

Integrating now shows that $|I_1^+| \lesssim t^{-\frac{3}{2}}$.

We now treat I_1^- . Since the stationary point is not explicitly calculable, some care is required. Because of the lower bound on ω'_r , we may find C depending only on c so that if $t < \delta_r C$ then $\Phi'_{r,t} < -1$ uniformly on $\text{supp } \chi$. Using this, we break into cases depending on the value of t :

Case 1: $t < \delta_r C$

Due to the lower bound on $|\Phi'_{r,t}|$, the phase is non-stationary and therefore the integral may be estimated similarly to I_1^+ .

Case 2: $t \geq \delta_r C$

In this regime, the phase may become stationary, however any stationary point will be non-degenerate because $(\Phi_{r,t}^-)' \geq 2$ on $\text{supp } \chi$ uniformly in r by the properties of ω_r . Indeed, because the second derivative is bounded below away from 0, we claim that we may always find some $\sigma_* \in \text{supp } \chi$ so that $|(\Phi_{r,t}^-)'(\sigma)| \geq 2|\sigma - \sigma_*|$ on $\text{supp } \chi$. If $(\Phi_{r,t}^-)'$ vanishes at some σ_* then this is immediate from the mean value theorem. Otherwise, we know that $\Phi_{r,t}^-$ is increasing on $[a, b] = \text{supp } \chi$ so we must have that either $(\Phi_{r,t}^-)'(a) > 0$ or $(\Phi_{r,t}^-)'(b) < 0$ if $\Phi_{r,t}^-$ does not vanish. In either case, the claim is easily seen to hold with $\sigma_* = a$ or b , respectively.

Splitting I_1^- as With this in mind, we write

$$I_1^- = \frac{1}{2it} \int_{|\sigma - \sigma_*| < t^{-\frac{1}{2}}} e^{it\Phi_{r,t}^-(\sigma)} b_-(\sigma; r) \chi(\sigma) d\sigma + \frac{1}{2it} \int_{|\sigma - \sigma_*| > t^{-\frac{1}{2}}} e^{it\Phi_{r,t}^-(\sigma)} b_-(\sigma; r) \chi(\sigma) d\sigma.$$

As before, the integrand of the first integral is bounded so by integrating by parts in the second we see that we need only estimate

$$t^{-2} \int_{|\sigma - \sigma_*| > t^{-\frac{1}{2}}} \left(\frac{b'_-(\sigma; r)}{(\Phi_{r,t}^-)'(\sigma)} - \frac{b_-(\sigma; r)(\Phi_{r,t}^-)''(\sigma)}{((\Phi_{r,t}^-)'(\sigma))^2} + \frac{b_-(\sigma; r)}{(\Phi_{r,t}^-)'(\sigma)} \chi'(\sigma) \right) d\sigma.$$

The term with χ' is easily seen to be admissible whereas the rest of the integral may be bounded by

$$\begin{aligned} & t^{-2} \int_{|\sigma - \sigma_*| > t^{-\frac{1}{2}}} \left(\frac{|b'_-(\sigma; r)|}{|\sigma - \sigma_*|} + \frac{|b_-(\sigma; r)| \sup |(\Phi_{r,t}^-)''(\sigma)|}{|\sigma - \sigma_*|^2} \right) \chi(\sigma) d\sigma \\ & \lesssim t^{-\frac{3}{2}} \int_0^\infty |b'_-(\sigma; r)| \chi(\sigma) d\sigma + t^{-2} \int_{|\sigma - \sigma_*| > t^{-\frac{1}{2}}} \frac{|b_-(\sigma; r)| \sup |\Phi_{r,t}^-''(\sigma)|}{|\sigma - \sigma_*|^2} d\sigma. \end{aligned}$$

The bounds in (4.5) show that the first integral is bounded by $t^{-\frac{3}{2}}$ whereas for the second we use that

$$|b_-(\sigma; r) \Phi_{r,t}^-''(\sigma)| \lesssim t^{-1} \sigma |\omega_r''(\sigma)| \lesssim \frac{\delta_r}{t} \lesssim 1$$

to conclude. This finishes the proof. \square

We are now ready to show that

Lemma 4.5. *We have that $|K_2(r, s; t)| \lesssim t^{-\frac{3}{2}}$.*

Proof. Since $\tilde{\chi}_k(\sigma^2 r)\chi_k(\sigma^2 s)$ only has non-empty support if $r > s$, it suffices to consider

$$\int_0^\infty e^{it\sigma^2} \frac{e(\sigma, r)}{r} \frac{e(\sigma, s)\chi_k(\sigma^2 s)}{s} \chi(\sigma) d\sigma$$

where as in Lemma 4.3 $\chi(\sigma) = \chi_c(\sigma)\tilde{\chi}_k(\sigma^2 r)$. By Corollary 2.14, on the support of χ , $e(\sigma, r) = e^{i\zeta_r(\sigma)}a_+(\sigma, r) + e^{-i\zeta_r(\sigma)}a_-(\sigma, r)$. Therefore, we need to estimate the integrals

$$(4.6) \quad I_\pm := \int_0^\infty e^{i(t\sigma^2 \pm \zeta_r)} \frac{a_\pm(\sigma, r)}{r} \frac{b_s(\sigma)}{s} \chi(\sigma) d\sigma$$

where $b_s(\sigma) := \chi_k(\sigma^2 s)e(\sigma, s)$.

We verify the conditions of Lemma 4.3. Proposition 2.13 shows that ζ_r satisfies the hypotheses of the lemma so we need only check that $a_{r,s}(\sigma) = \frac{a_-(\sigma)}{r} \frac{b_s(\sigma)}{s}$ satisfies the hypotheses as well, uniformly in s . From Corollary 2.14 we obtain

$$|a_-(\sigma, r)| \lesssim 1, \quad |a'_-(\sigma, r)| \lesssim \sigma^{-1}, \quad |a''_-(\sigma, r)| \lesssim r$$

and from Corollary 2.17 that

$$|b_s(\sigma)| \lesssim s\sigma^2, \quad |b'_s(\sigma)| \lesssim s[\sigma^2\chi_{\frac{1}{2}}(\sigma^2 s) + \chi_{\frac{1}{2},k}(\sigma^2 s)], \quad |b''_s(\sigma)| \lesssim s[\sigma^2\chi_{\frac{1}{2}}(\sigma^2 s) + \sigma^{-2}\chi_{\frac{1}{2},k}(\sigma^2 s)],$$

where $\chi_{\frac{1}{2},k} = (\tilde{\chi}_{\frac{1}{2}} \cdot \chi_k)(\sigma^2 s)$. Clearly $|a_{r,s}(\sigma)| \lesssim \frac{\sigma^2}{r}$ and furthermore

$$|a'_{r,s}(\sigma)| \lesssim r^{-1}[\sigma^2\chi_{\frac{1}{2}}(\sigma^2 s) + \chi_{\frac{1}{2},k}(\sigma^2 s)] + \frac{\sigma}{r} \lesssim \sigma^2$$

since $r^{-1} \lesssim \sigma^2$ on the support of χ . Proceeding, one may also check that

$$|a''_{r,s}(\sigma)| \lesssim \sigma\chi_{\frac{1}{2}}(\sigma^2 s) + \chi_{\frac{1}{2},k}(\sigma^2 s)$$

It now follows from the computation

$$\int_0^\infty \sigma^{-1}\chi_{\frac{1}{2},k}(\sigma^2 s)\chi d\sigma \leq \log(2k)/2$$

that $\int_0^\infty \sigma^{-1}(|a''_{r,s}(\sigma)| + r|a'_{r,s}(\sigma)|) d\sigma \lesssim 1$ so we conclude from Lemma 4.3 that $I_1 = O(t^{-\frac{3}{2}})$. \square

We next prove that

Lemma 4.6. $|K_3(r, s; t)| \lesssim t^{-\frac{3}{2}}$.

Proof. By Corollary 2.14, in this σ regime, $e(\sigma, r)e(\sigma, s)$ can be written as a sum of the terms

$$(4.7) \quad e^{-i(\zeta_r(\sigma) \pm \zeta_s(\sigma))} a_-(\sigma, r) a_\pm(\sigma, s), \quad e^{i(\zeta_r(\sigma) \pm \zeta_s(\sigma))} a_+(\sigma, r) a_\pm(\sigma, s)$$

Therefore, it suffices to bound

$$(4.8) \quad \int_0^\infty e^{it\sigma^2 - i(\zeta_r(\sigma) \pm \zeta_s(\sigma))} \frac{a_-(\sigma, r)}{r} \frac{a_\pm(\sigma, s)\tilde{\chi}_k(\sigma^2 s)}{s} \chi(\sigma) d\sigma$$

$$(4.9) \quad \int_0^\infty e^{it\sigma^2 + i(\zeta_r(\sigma) \pm \zeta_s(\sigma))} \frac{a_+(\sigma, r)}{r} \frac{a_\pm(\sigma, s) \tilde{\chi}_k(\sigma^2 s)}{s} \chi(\sigma) d\sigma,$$

where $\chi(\sigma) := \chi_c(\sigma) \tilde{\chi}_k(\sigma^2 r)$. In order to apply Lemma 4.3, we must verify that the phases $\zeta_r + \zeta_s$ and $\zeta_r - \zeta_s$ and the amplitude $a_{r,s}(\sigma) = \frac{a_+(\sigma, s)}{r} \frac{a_+(\sigma, s) \tilde{\chi}_k(\sigma^2 s)}{s}$ satisfy the hypotheses of the lemma (the latter being sufficient because a_+ and a_- obey the same bounds). When $r = s$, the phase $\zeta_r - \zeta_s$ degenerates to 0. Ignoring this easily treated case, the conditions on the phases are satisfied by Proposition 2.13 so we consider the amplitude $a_{r,s}(\sigma)$. From Corollary 2.14, we see that

$$|a_{r,s}(\sigma)| \lesssim \frac{\tilde{\chi}_k(\sigma^2 s)}{rs} \lesssim \frac{\sigma^2}{r}$$

and also that

$$|a'_{r,s}(\sigma)| \lesssim \frac{\sigma^{-1} \tilde{\chi}_k(\sigma^2 s)}{rs} \lesssim \frac{1}{rs^{\frac{1}{2}}}$$

which is indeed less than σ^2 on the domain in question. Furthermore, we have that

$$|a''_{r,s}(\sigma)| \lesssim r^{-1}$$

from which one can easily check that $\int_0^\infty \sigma^{-1} (|a''_{r,s}(\sigma)| + r|a'_{r,s}(\sigma)|) \chi(\sigma) d\sigma \lesssim 1$.

Applying Lemma 4.3 now completes the proof. \square

Proof of Proposition 4.1. Combining the bounds for K_1 , K_2 and K_3 we obtain the statement. \square

4.2. Estimate of $K_t^h(r, s)$. We will prove the following Proposition.

Proposition 4.7. *Let $c < 1$ and $t \geq 1$ then we have $\sup_{r,s} |K_t^h(r, s)| \lesssim t^{-\frac{3}{2}}$.*

Similar to previous section we will prove Proposition 4.7 with a series of lemmas. We let $k \geq 4$ and write

$$(4.10) \quad \begin{aligned} K_t^h(r, s) &= \frac{1}{rs} \int_0^\infty e^{it\sigma^2} \tilde{\chi}_c(\sigma) \chi_k(\sigma r) \chi_k(\sigma s) e(\sigma, r) e(\sigma, s) d\sigma \\ &\quad + \frac{1}{rs} \int_0^\infty e^{it\sigma^2} \tilde{\chi}_c(\sigma) [\chi_k(\sigma r) \tilde{\chi}_k(\sigma s) + \tilde{\chi}_k(\sigma r) \chi_k(\sigma s)] e(\sigma, r) e(\sigma, s) d\sigma \\ &\quad + \frac{1}{rs} \int_0^\infty e^{it\sigma^2} \tilde{\chi}_k(\sigma) \tilde{\chi}_k(\sigma s) \tilde{\chi}_c(\sigma^2 r) e(\sigma, r) e(\sigma, s) d\sigma \\ &= \tilde{K}_1(r, s; t) + \tilde{K}_2(r, s; t) + \tilde{K}_3(r, s; t). \end{aligned}$$

We start estimating the first term.

Lemma 4.8. $|\tilde{K}_1(r, s; t)| \lesssim t^{-\frac{3}{2}}$.

Proof. Let $a(\sigma; r, s) = (rs)^{-1} \tilde{\chi}_c(\sigma) \chi_k(\sigma r) \chi_k(\sigma s) e(\sigma, r) e(\sigma, s)$, then using (3.2), we have

$$(4.11) \quad a(\sigma; r, s) = \tilde{\chi}_c(\sigma) \chi_k(\sigma r) \chi_k(\sigma s) \sigma [e^{\frac{\pi}{\sigma}} - 1]^{-1} (1 + O_2(\sigma s) + O_2(\sigma r)).$$

Therefore, we have

$$|a(\sigma; r, s)| \lesssim \sigma^2 \tilde{\chi}_c(\sigma) \chi_k(\sigma r) \chi_k(\sigma s).$$

Moreover, since $\chi_c(\sigma r) = O_\infty(\sigma^0)$ we in fact, have $a(\sigma; r, s) = \tilde{\chi}_c(\sigma) \chi_c(\sigma r) O_2(\sigma^2)$. Hence, by twice integration by parts, we obtain

$$(4.12) \quad \left| \int_0^\infty e^{it\sigma^2} a(\sigma; r, s) d\sigma \right| \lesssim t^{-2} \int_0^\infty \left| \left(\frac{1}{\sigma} \left(\frac{a(\sigma; r, s)}{\sigma} \right)' \right)' \right| d\sigma \lesssim t^{-2} \int_c^\infty \sigma^{-2} d\sigma \lesssim t^{-2}.$$

We obtain no boundary terms since the support of the integral is away from both zero and infinity. \square

We next prove the following oscillatory lemma which will be useful to estimate the contributions of the rest of the terms.

Lemma 4.9. *Let $a(\sigma) = \tilde{\chi}_c(\sigma) O_2(\sigma)$. Then one has*

$$I(r, s; t) = \int_0^\infty e^{it\sigma^2 \pm i\varphi_u(\sigma)} \tilde{\chi}_k(\sigma r) a(\sigma) d\sigma \lesssim t^{-\frac{3}{2}} \max\{u, r\}$$

provided that the condition (4.13) holds for $\varphi_u(\sigma)$ within the support of the integral:

$$(4.13) \quad \varphi'_u(\sigma) \sim u, \quad \varphi''_u(\sigma) < 0, \quad |\varphi''_u(\sigma)| \lesssim \sigma^{-2} u.$$

Proof. As in the proof of Lemma 4.3, we start with an integration by parts and write

$$\begin{aligned} I^\pm(r, s; t) &= \frac{1}{2it} \int_0^\infty e^{it\sigma^2 \pm i\varphi_u(\sigma)} [b_1(\sigma; r) + b_2(\sigma; r, u)] d\sigma \\ &=: I_1^\pm + I_2^\pm \end{aligned}$$

with

$$b_1(\sigma; r) := \sigma^{-1} (\tilde{\chi}_k(\sigma r) a(\sigma))' - \sigma^{-2} \tilde{\chi}_k(\sigma r) a(\sigma), \quad b_2(\sigma; r, u) := \pm i \sigma^{-1} \tilde{\chi}_k(\sigma r) a(\sigma) \varphi'_u(\sigma).$$

We apply another integration by parts to I_1^\pm to bound it by

$$t^{-2} \int_0^\infty |(\sigma^{-1} b_1(\sigma; r))'| + \sigma^{-1} |b_1(\sigma; r) \varphi'_r(\sigma)| d\sigma.$$

We have, $\varphi'_u(\sigma) \lesssim u$, $|b_1| \lesssim \sigma^{-1}$ and $|(\sigma^{-1}b_1)'| \lesssim \tilde{\chi}(\sigma)\sigma^{-3} \lesssim \sigma^{-2}r$. Therefore, $I_1^\pm \lesssim t^{-2} \max\{r, u\}$.

We next focus on I_2^\pm . We let $\Phi_\pm(\sigma) = \sigma^2 \pm t^{-1}\varphi_u(\sigma)$. Note that the conditions on φ_u is arranged so that only $\Phi_-(\sigma)$ might have a critical point. In fact, as $\varphi''_u(\sigma) < 0$, for each fixed r , if this critical point exist then it must be non-degenerate in the support of $\tilde{\chi}_k(\sigma r)$. Furthermore, since $\varphi'_u(\sigma)$ is a decreasing function with respect to σ , as opposed to 2σ , there is always $\sigma^* \sim u/t$ such that $|\Phi'_-(\sigma)| \gtrsim |\sigma - \sigma^*|$.

Having these in mind, we first focus on I_2^- and divide I_2^- as

$$I_2^- = (2it)^{-1} \int_{|\sigma - \sigma_*| \leq t^{-\frac{1}{2}}} e^{it\Phi_\pm(\sigma)} b_2(\sigma; r, u) d\sigma + (2it)^{-1} \int_{|\sigma - \sigma_*| > t^{-\frac{1}{2}}} e^{it\Phi_\pm(\sigma)} b_2(\sigma; r, u) d\sigma.$$

We have $|b_2(\sigma; r, u)| \lesssim u$, therefore the first term in I_2^- is bounded by $t^{-\frac{3}{2}}u$. We apply another integration by parts to bound the second term in I_2^- by

$$(4.14) \quad t^{-2} \int_{|\sigma - \sigma_*| > t^{-\frac{1}{2}}} \frac{|b'_2(\sigma; r, u)|}{|\Phi'_-(\sigma)|} + \frac{|b_2(\sigma; r, u)||\Phi''_-(\sigma)|}{|\Phi'_-(\sigma)|^2} d\sigma$$

where we omit the boundary term since it will be simply bounded by (4.14). Note that, we have $|b'_2(\sigma; r, u)| \lesssim \sigma^{-1}u$. Therefore,

$$\frac{|b'_2(\sigma; r, u)|}{|\Phi'_-(\sigma)|} \lesssim \frac{u}{\sigma|\sigma - \sigma_*|} \lesssim \frac{u}{\sigma^2} + \frac{u}{|\sigma - \sigma_*|^2}.$$

Furthermore, as $\Phi''(\sigma) = 2 - t^{-1}\varphi''_u$ and $|\varphi''_u| \lesssim \sigma^{-2}u$

$$\frac{|b_2(\sigma; r, u)||\Phi''_-(\sigma)|}{|\Phi'_-(\sigma)|^2} \lesssim \frac{u}{|\sigma - \sigma_*|^2} + \frac{u^2}{|\sigma - \sigma_*|^2\sigma^2 t} \lesssim \frac{u}{|\sigma - \sigma_*|^2} + \frac{u\sigma_*}{|\sigma - \sigma_*|^2\sigma^2}.$$

Note that if $\sigma_* \leq 1$, then

$$\frac{|b_2(\sigma; r, u)||\Phi''_-(\sigma)|}{|\Phi'_-(\sigma)|^2} \lesssim \frac{us}{|\sigma - \sigma_*|^2}.$$

as $\sigma > c$. On the other hand if $\sigma_* \geq 1$ and $|\sigma - \sigma_*| \leq \sigma_*/2$ then $\sigma \sim \sigma_*$ and

$$\frac{u\sigma_*}{|\sigma - \sigma_*|^2\sigma^2} \lesssim \frac{u}{\sigma_*|\sigma - \sigma_*|^2} \lesssim \frac{u}{|\sigma - \sigma_*|^2}.$$

If $\sigma_* \geq 1$ and $|\sigma - \sigma_*| \geq \sigma_*/2$ then

$$\frac{u\sigma_*}{|\sigma - \sigma_*|^2\sigma^2} \lesssim \frac{u}{\sigma^2\sigma_*} \lesssim \frac{u}{\sigma^2}.$$

Therefore, the integrand in (4.14) is bounded by $u\sigma^{-2} + u|\sigma - \sigma_*|^{-2}$. The first term is integrable away from zero and the integration of the second one in $|\sigma - \sigma_*| \geq t^{-\frac{1}{2}}$ is bounded by $ut^{\frac{1}{2}}$. Therefore, we have $I_2^- \lesssim ut^{-3/2}$.

Finally, we consider I_2^+ where we have no critical points. We apply another integration by parts to this integral to see

$$I_2^+ \lesssim t^{-2} \int_0^\infty \frac{|b'_2(\sigma; r, u)|}{|\Phi'_+(\sigma)|} + \frac{|b_2(\sigma; r, u)| |\Phi''_+(\sigma)|}{|\Phi'_+(\sigma)|^2} d\sigma \lesssim t^{-2} u$$

since, $|b'_2(\sigma; r, u)| \lesssim \sigma^{-1} u$, $|\Phi'_+(\sigma)|^{-1} \lesssim |\sigma + t^{-1} \varphi'_u(\sigma)|^{-1} \lesssim \sigma^{-1}$ and $|b_2(\sigma; r, u)| \lesssim u$, $|\Phi'_+(\sigma)|^{-1} \lesssim |t^{-1} \varphi'_u(\sigma)| \lesssim u/t$.

□

We continue estimating the term $\tilde{K}_2(r, s; t)$.

Lemma 4.10. $|\tilde{K}_2(r, s; t)| \lesssim t^{-\frac{3}{2}}$.

Proof. Taking into account the symmetry in $\tilde{K}_2(r, s; t)$ with respect to r and s , we concentrate on estimating the following expression:

$$(rs)^{-1} \int_0^\infty e^{it\sigma^2} \tilde{\chi}_c(\sigma) \tilde{\chi}_k(\sigma r) \chi_k(\sigma s) e(\sigma, r) e(\sigma, s) d\sigma.$$

Using Proposition 3.1, we compute

$$(4.15) \quad \begin{aligned} \tilde{\chi}_c(\sigma) \tilde{\chi}_k(\sigma r) e(\sigma, r) \chi_k(\sigma s) e(\sigma, s) &= e^{i(\sigma r - \sigma^{-1} \log(2\sigma r))} e^{i\theta(\sigma)} \tilde{\chi}_k(\sigma r) \tilde{\chi}_c(\sigma) \chi_k(\sigma s) e(\sigma, s) \\ &\quad + e^{-i(\sigma r - \sigma^{-1} \log(2\sigma r))} e^{-i\theta(\sigma)} \tilde{\chi}_k(\sigma r) \tilde{\chi}_c(\sigma) \chi_k(\sigma s) e(\sigma, s) \\ &\quad + \mathcal{E}(\sigma, r) \chi_k(\sigma s) e(\sigma, s). \end{aligned}$$

We first focus on the first two terms in (4.15). Taking $u = r$ and applying Lemma 4.9 to $\varphi_r(\sigma) = \sigma r - \sigma^{-1} \log(2\sigma r)$ and $a(\sigma) = \tilde{\chi}_c(\sigma) e^{\pm i\theta(\sigma)} \chi_k(\sigma s) e(\sigma, s)$, we proceed to estimate

$$(4.16) \quad (rs)^{-1} \int_0^\infty e^{it\sigma^2 \pm i\varphi_r(\sigma)} \tilde{\chi}_c(\sigma) \tilde{\chi}_k(\sigma r) a(\sigma) d\sigma.$$

Note that $\varphi'_r(\sigma) = r + \sigma^{-2} [\log(2\sigma r) - 1]$. Within the domain of $\tilde{\chi}_k(\sigma r)$, it holds that $\log(2\sigma r) > 1$, leading to the inequality $r \leq \varphi'_r(\sigma) \leq (1 + c^{-1})r$. Additionally, we find that $\varphi''_r(\sigma) = -\sigma^3 [2\log(2\sigma r) + 3] < 0$. Therefore, in the support of $\tilde{\chi}_k(\sigma r)$, it's evident that $|\varphi''_r(\sigma)| \lesssim \sigma^{-2} r$. Utilizing (3.2), along with the fact that $|\theta'| \lesssim \sigma^{-2}$ and $|\theta''| \lesssim \sigma^{-3}$, we observe that $a(\sigma) = s \tilde{\chi}_c(\sigma) O_2(\sigma)$. Hence, by applying Lemma 4.9, we can conclude that $|(4.16)| \lesssim t^{-\frac{3}{2}}$.

We next consider the last term in the expansion of $\tilde{\chi}_c(\sigma) \tilde{\chi}_k(\sigma r) e(\sigma, r) \chi_k(\sigma s) e(\sigma, s)$. That is we need to bound

$$(4.17) \quad (rs)^{-1} \int_0^\infty e^{it\sigma^2} \tilde{\chi}_k(\sigma s) e(\sigma, s) \mathcal{E}(\sigma, r) d\sigma$$

By (3.2) and (3.4) we have

$$(4.18) \quad |\partial_\sigma^j \{\mathcal{E}_\pm(\sigma, r)\chi_k(\sigma s)e(\sigma, s)\}| \lesssim \sigma^{2-j} r s \tilde{\chi}_c(\sigma)\chi_k(\sigma s).$$

Hence, by integration by parts as in (4.12), we bound (4.17) by t^{-2} . This finishes the proof. \square

We finally prove

Lemma 4.11. $|\tilde{K}_3(r, s; t)| \lesssim t^{-\frac{3}{2}}$.

Proof. We start computing the integrand in $\tilde{K}_3(r, s; t)$. Note that by Proposition 3.1, we may write $\tilde{\chi}_c(\sigma r)e(\sigma, r)\tilde{\chi}_c(\sigma s)e(\sigma, s)$ as the sum of the following terms

$$(4.19) \quad \begin{aligned} & e^{\pm i([\sigma r - \sigma^{-1} \log(2\sigma r)] + [\sigma s - \sigma^{-1} \log(2\sigma s)])} e^{\pm 2i\theta(\sigma)} \tilde{\chi}_c(\sigma r)\tilde{\chi}_k(\sigma)\tilde{\chi}_k(\sigma s), \\ & e^{\pm i([\sigma r - \sigma^{-1} \log(2\sigma r)] - [\sigma s - \sigma^{-1} \log(2\sigma s)])} \tilde{\chi}_c(\sigma r)\tilde{\chi}_k(\sigma)\tilde{\chi}_k(\sigma s), \\ & e^{\pm i(\sigma s - \sigma^{-1} \log(2\sigma s))} \tilde{\chi}_k(\sigma s) e^{\pm i\theta(\sigma)} \mathcal{E}(\sigma, r), \\ & e^{\pm i(\sigma r - \sigma^{-1} \log(2\sigma r))} \tilde{\chi}_k(\sigma r) e^{\pm i\theta(\sigma)} \mathcal{E}(\sigma, s), \\ & \mathcal{E}(\sigma, s)\mathcal{E}(\sigma, r). \end{aligned}$$

We first consider the last three terms in (4.19). Since the third and fourth terms are symmetric, it will be enough to bound

$$(4.20) \quad (rs)^{-1} \int_0^\infty e^{it\sigma^2 \pm \varphi_r(\sigma)} \tilde{\chi}_c(\sigma)\tilde{\chi}_k(\sigma r) a_\pm(\sigma) d\sigma + (rs)^{-1} \int_0^\infty e^{it\sigma^2} \mathcal{E}(\sigma, r)\mathcal{E}(\sigma, s) d\sigma,$$

where $a_\pm(\sigma) = e^{\pm i\theta(\sigma)} \mathcal{E}_\pm(\sigma, s)$ and $\varphi_r(\sigma) = \sigma r - \sigma^{-1} \log(2\sigma r)$. By previous lemma $\varphi_r(\sigma)$ satisfies the conditions in (4.13). Moreover, $a_\pm(\sigma) = s\tilde{\chi}_c(\sigma)O_2(\sigma)$. Hence, by Lemma 4.9 we conclude that the first term in (4.20) is bounded by $t^{-\frac{3}{2}}$. For the second term in (4.20), we have

$$(4.21) \quad |\mathcal{E}(\sigma, r)\mathcal{E}(\sigma, s)| \lesssim 1, |\partial_\sigma^j \{\mathcal{E}(\sigma, r)\mathcal{E}(\sigma, s)\}| \lesssim \sigma^{2-j} r s, \quad j = 1, 2.$$

Therefore, by integration by parts twice we can bound the second term in (4.20) by t^{-2} .

We finally focus on the first two terms in (4.19). For the first term, we let $\varphi_{r+s}(\sigma) = [\sigma r + \sigma^{-1} \log(2\sigma r)] + [\sigma s - \sigma^{-1} \log(2\sigma s)]$ and estimate the integral

$$(4.22) \quad (rs)^{-1} \int_0^\infty e^{it\sigma^2 \pm i\varphi_{r+s}(\sigma)} \tilde{\chi}_c(\sigma)\tilde{\chi}_k(\sigma r)\tilde{\chi}_k(\sigma s) e^{\pm 2i\theta(\sigma)} d\sigma.$$

Note that (4.22) is symmetric with respect to r and s . Therefore, without loss of generality, we assume $r \geq s$ and use Lemma 4.9 for $a_\pm(\sigma) = \tilde{\chi}_c(\sigma)\tilde{\chi}_k(\sigma s) e^{\pm 2i\theta(\sigma)}$. While it is evident that $a_\pm(\sigma) = s\tilde{\chi}_c(\sigma)O(\sigma)$, we still need to demonstrate that φ_{r+s} satisfies the conditions in (4.13).

We calculate $\varphi'_{r+s}(\sigma) = (r+s) + \sigma^{-2} [\log(2\sigma r) + \log(2\sigma s) - 2]$. Thus, within the domain of $\tilde{\chi}_c(\sigma s)\tilde{\chi}_c(\sigma r)$, we have $r+s \leq \varphi'_{r+s} \leq (r+s)(1+1/c)$. It is also possible to compute

$\varphi''_{r+s} = -\sigma^3 [2 \log(2\sigma r) + 2 \log(2\sigma s) + 6] < 0$. Noting that $r + s \leq 2r$, we can deduce from Lemma 4.9 that $|(4.22)| \lesssim t^{-\frac{3}{2}}$.

Finally, we consider the second term in (4.19) and estimate

$$(4.23) \quad (rs)^{-1} \int_0^\infty e^{it\sigma^2 \pm i\varphi_{r-s}(\sigma)} \tilde{\chi}_k(\sigma r) a(\sigma) d\sigma$$

where $\varphi_{r-s}(\sigma) = [\sigma r + \sigma^{-1} \log(2\sigma r)] - [\sigma s - \sigma^{-1} \log(2\sigma s)]$ and $a(\sigma) = \tilde{\chi}_c(\sigma) \tilde{\chi}_k(\sigma s)$. We compute $\varphi'_{r-s}(\sigma) = (r-s) + \sigma^{-2} \log(r/s)$. Since we can assume $r \geq s$ due to the symmetry, we immediately have $\varphi'_{r-s}(\sigma) \geq r-s$. Furthermore, by the mean value theorem, we have $0 < \log(\sigma r) - \log(\sigma s) \lesssim (r-s)s^{-1}$ and

$$\sigma^{-2} \log(r/s) \lesssim \frac{(r-s)}{\sigma^2 s} \lesssim c^{-1}(r-s)$$

in the support of $\sigma s \geq k$. Hence, $\varphi'_{r-s}(\sigma) \sim r-s$ and $\varphi''_{r-s}(\sigma) \lesssim \sigma^{-2}(r-s)$. Moreover, $a(\sigma) = s \tilde{\chi}_c(\sigma) O_2(\sigma)$ and therefore, by Lemma 4.9 we have $|(4.23)| \lesssim t^{-\frac{3}{2}}$. □

Proof of Proposition 4.7. Combining the bounds for \tilde{K}_1 , \tilde{K}_2 and \tilde{K}_3 , we obtain the statement. □

APPENDIX A. KERNEL OF THE COULOMB EVOLUTION

The analysis above stems from the following explicit representation of the time evolution of $H_{0,q}$ for $q > 0$:

$$(A.1) \quad [e^{itH_{0,q}} f](r) = -\frac{q}{2r} \int_0^\infty \int_0^\infty e^{it\sigma^2} M_{\frac{iq}{2\sigma}, \frac{1}{2}}(2i\sigma r) M_{\frac{iq}{2\sigma}, \frac{1}{2}}(2i\sigma s) s f(s) \sigma^{-1} [e^{\frac{q\pi}{\sigma}} - 1]^{-1} d\sigma ds$$

where $M_{\frac{iq}{2\sigma}, \frac{1}{2}}(\cdot)$ is the Whittaker-M function (see [8, Ch 13]). Upon substituting σ with $q\sigma$, equation (A.1) transforms into:

$$\frac{q}{2r} \int_0^\infty \int_0^\infty e^{itq^2\sigma^2} e(q\sigma, r) e(q\sigma, s) s f(s) \sigma^{-1} [e^{\frac{\pi}{\sigma}} - 1]^{-1} d\sigma ds.$$

Here, the function $e(q\sigma, r)$ is defined as:

$$e(q\sigma, r) := -i\sigma^{-\frac{1}{2}} [e^{\frac{\pi}{\sigma}} - 1]^{-\frac{1}{2}} M_{\frac{i}{2\sigma}, \frac{1}{2}}(2iq\sigma r).$$

This representation is obtained by diagonalizing $rH_{0,q}r^{-1} = -\frac{d^2}{dr^2} + \frac{q}{r}$ via the *distorted Fourier transform*. The purpose of this section is to explain the proof of (A.1), namely

Theorem A.1. *For all $f \in rC_{0,\text{rad}}^\infty(\mathbb{R}^3)$, the equality (A.1) holds.*

A.1. Review of Weyl-Titchmarsh theory. We begin by briefly recalling some of the basic spectral theory of half-line Schrödinger operators with a regular left endpoint. In particular, we summarize the construction of the distorted Fourier transform. This theory is well-known and more details may be found in: [4], [5, Ch.9], [10, Sect. XIII.5], [11, Ch. 2], [15], [18], [21, Ch. 10], [25], [27], [28, Ch. 2], [30, Ch. VI], [34, Ch. 6], [35, Ch.X], [41, Chs. II, III], [43, Sects. 7–10] While our potential is not regular at 0 due to the $\frac{1}{r}$ singularity, the theory of singular potentials is developed in parallel to the regular case.

Consider the symmetric Schrödinger operator

$$H = -\frac{d^2}{dx^2} + V(x), \quad V = \overline{V} \in L^1_{\text{loc}}(\mathbb{R}_+)$$

with domain $\mathcal{D}(H) = C_0^2(\mathbb{R}_+)$. We assume that $V \in L^1(0,1)$ and that it is *limit point* at ∞ , that is, for any $z \in \mathbb{C} \setminus \mathbb{R}$, the space of solutions to $Hf = zf$ that are L^2 at ∞ is at most 1-dimensional. For instance, it is sufficient (but by no means necessary) to assume that V is bounded at ∞ . For $\alpha \in [0, \pi]$, let H_α be the self-adjoint extension of H with the domain

$$\mathcal{D}_\alpha := \{g \in H^2(\mathbb{R}_+) \mid \sin(\alpha)g'(0) + \cos(\alpha)g(0) = 0\}.$$

We first define $\phi_\alpha(z, x)$ and $\theta_\alpha(z, x)$ as the fundamental system of solutions to $H_\alpha f = -z^2 f$, for $z \in \mathbb{C}$, that satisfy

$$(A.2) \quad \phi_\alpha(z, 0) = -\theta'_\alpha(z, 0) = -\sin(\alpha), \quad \phi'_\alpha(z, 0) = \theta_\alpha(z, 0) = \cos(\alpha), \quad W(\phi(z, \cdot), \theta(z, \cdot)) = 1.$$

Because V is L^1 near 0, the existence of ϕ_α and θ_α is assured by Picard iteration as is their analyticity as functions of z . Furthermore, they are real-valued for $z^2 \in \mathbb{R}$.

We next define a *Weyl solution* $\psi_\alpha(z, \cdot)$ near infinity (or zero) to be a non-zero solution to $H_\alpha f = -z^2 f$ that is L^2 near infinity (or zero). We note that, as long as V is continuous and real valued in $(0, \infty)$, there exist at least one Weyl solution near infinity and at least one Weyl solution near zero, see Theorem X.6 of [35]. Since H is in the limit point case at infinity the Weyl solution near infinity is unique (up to scaling), whereas because H is in the limit circle case at zero, all solutions are Weyl solutions near zero. Hence, we can uniquely characterize the Weyl solution at infinity as

$$\psi_\alpha(x, z) = \theta_\alpha(x, z) + m(z)\phi_\alpha(x, z)$$

where $m(z) = W(\theta(z, \cdot), \psi(z, \cdot))$, is the Weyl- m function, which is analytic for $z^2 \in \mathbb{C} \setminus \mathbb{R}$. Note that this representation is possible because the θ_α coefficient of ψ_α cannot vanish or ψ_α would be an eigenfunction for non-real z^2 .

The significance of the Weyl solution is that it allows us write the resolvent kernel or Green's function via

$$(A.3) \quad (H_\alpha + z^2)^{-1} f(x) = \int_0^\infty [\phi(x, z)\psi(y, z)\chi_{[0 < x < y]} + \phi(y, z)\psi(x, z)\chi_{[x > y > 0]}] f(y) dy.$$

With these objects in hand, we are ready to define the distorted Fourier transform:

Proposition A.2. *For $f \in C_0([0, \infty))$ let*

$$[U_\alpha f](\lambda) = \int_0^\infty f(x)\phi_\alpha(x, \lambda) dx.$$

Then we have the following Plancharel theorem

$$\|f\|_{L^2(\mathbb{R}_+)} = \|U_\alpha f\|_{L^2(\mathbb{R}, \rho)}$$

and inversion formula

$$f(x) = \lim_{b \rightarrow \infty} \int_{-b}^b [U_\alpha f](\lambda)\phi_\alpha(x, \lambda) \rho(d\lambda).$$

In particular, for any $F \in C(\mathbb{R})$ and $f \in C_0^\infty([0, \infty))$, we have

$$(A.4) \quad [F(H_\alpha)f](\cdot) = \int_{\sigma(H_\alpha)} \int_0^\infty F(\sigma^2)\phi_\alpha(\sigma, \cdot)\phi_\alpha(\sigma, x)f(x) dx \rho(d\sigma).$$

We refer to ϕ_α as the distorted Fourier basis and to ρ as the associated spectral measure. The proof comes from plugging (A.3) into Stone's formula. Recall that Stone's formula is given for $\lambda^2 \in \mathbb{R}$ and $f \in C_0^\infty([0, \infty))$ as

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\pi i} \int_a^b \langle [R(\lambda^2 + i\varepsilon) - R(\lambda^2 - i\varepsilon)]f, f \rangle \lambda d\lambda = \langle [E(a, b) + \frac{1}{2}(E(\{a\}) + E(\{b\}))]f, f \rangle,$$

where $-\infty \leq a \leq b \leq \infty$, $E(\cdot)$ is the spectral resolution of H_α , $R(z) := (H_\alpha - z)^{-1}$ is the resolvent operator, and we adopt the convention $E(\{\pm\infty\}) = 0$. The result then follows from the fact that $\rho(d\lambda)$ is recoverable via the weak-* limit as $\varepsilon \rightarrow 0$ of

$$\frac{\lambda}{\pi i} [m(\lambda^2 + i\varepsilon) - m(\lambda^2 - i\varepsilon)] d\lambda.$$

A.2. Proof of Theorem A.1. This section closely follows [18], which in turn relies on the idea of [22] to determine the spectral measure via Stone's formula. Let \mathcal{L}_q be the half-line Schrödinger operator that is unitarily equivalent to $H_{0,q}$, which we recall is the restriction of the Coulomb Hamiltonian H to the radial sector. In order to apply the above scheme to \mathcal{L}_q , we must first make sense of it as a self-adjoint operator. First, recall the following simple consequence of the Kato-Rellich theorem:

Lemma A.3 (Theorem X.15 in [35]). *Suppose that $V : \mathbb{R}^3 \rightarrow \mathbb{R}$ is equal to $V_1 + V_2$ where $V_1 \in L^2(\mathbb{R}^3)$ and $V_2 \in L^\infty(\mathbb{R}^3)$. Then $-\Delta + V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$ and self-adjoint on $H^2(\mathbb{R}^3)$.*

Clearly then our Hamiltonian H is a self-adjoint operator with domain $H^2(\mathbb{R}^3)$. Recall that \mathcal{L}_q is the half-line Schrödinger operator given by conjugating $H_{0,q}$ by r . Thus, it is automatically self-adjoint on the domain $rH_{\text{rad}}^2(\mathbb{R}^3)$ (where we regard functions on $H_{\text{rad}}^2(\mathbb{R}^3)$ as functions on \mathbb{R}_+). In particular, for $g \in \mathcal{D}(\mathcal{L})$ the function $\frac{g(r)}{r}$ is continuous at $r = 0$.

To compute the resolvent of $\mathcal{L}_q = \mathcal{L}$, first observe that a fundamental system of solutions to $\mathcal{L}f = -z^2 f$ for $\Re z^2 > 0$ is given by the Whittaker functions [8, 13.14]

$$M_{-\frac{q}{2z}, \frac{1}{2}}(2zr), \quad W_{-\frac{q}{2z}, \frac{1}{2}}(2zr).$$

These are solutions of Whittaker's equation

$$W''(\omega) + \left(-\frac{1}{4} + \frac{\kappa}{\omega} + \frac{\frac{1}{4} - \mu^2}{\omega^2} \right) W(\omega) = 0$$

which is related to $\mathcal{L}f = -z^2 f$ via $r = \frac{\omega}{2z}$ for $\kappa = -\frac{q}{2z}$ and $\mu = \frac{1}{2}$. By [8, (13.14.6)], we have

$$(A.5) \quad \frac{M_{-\frac{q}{2z}, \frac{1}{2}}(2zr)}{2z} = r e^{-rz} \left[1 + \sum_{s=1}^{\infty} \frac{(q+2z)(q/2+2z) \cdots (q/s+2z)}{s!} r^s \right]$$

and thus $\phi(z, r) := (2z)^{-1} M_{-\frac{q}{2z}, \frac{1}{2}}(2zr)$ is the unique solution satisfying the boundary condition for $\mathcal{D}(\mathcal{L})$ which is normalized so that $\phi'(0, z) = 1$. It is real analytic for $z^2 \leq 0$. Furthermore, by [8, (13.14.26)] we compute the Wronskian as

$$W[\phi(z, r), W_{-\frac{q}{2z}, \frac{1}{2}}(2zr)] = W[M_{-\frac{q}{2z}, \frac{1}{2}}(\cdot), W_{-\frac{q}{2z}, \frac{1}{2}}(\cdot)] = -\frac{1}{\Gamma(1 + \frac{q}{2z})} = -\frac{2z}{q\Gamma(q/(2z))}.$$

so we set $\psi(z, r) := -\frac{q}{2z}\Gamma(q/(2z))W_{-\frac{q}{2z}, \frac{1}{2}}(2zr)$ to ensure that $W[\phi, \psi] = 1$. This gives us the following representation of the resolvent of \mathcal{L} :

Proposition A.4. *For $\Re z > 0$, the resolvent kernel of \mathcal{L} is given by*

$$(\mathcal{L} + z^2)^{-1}(r, s) = \begin{cases} \phi(z, r)\psi(z, s), & 0 < r \leq s \\ \psi(z, r)\phi(z, s), & 0 < s \leq r \end{cases}.$$

Proof. This follows from the form of the resolvent of a Sturm-Liouville operator and the fact that ϕ is the unique solution satisfying the boundary condition of $\mathcal{D}(\mathcal{L})$. \square

In particular, if we let $-z^2 = \sigma^2 \pm i\varepsilon$ for $\sigma^2 \geq 0$, then we obtain

$$(\mathcal{L} - (\sigma^2 \pm i0))^{-1}(r, s) = \begin{cases} \phi(\pm i\sigma, r)\psi(\pm i\sigma, s), & 0 < r \leq s \\ \psi(\pm i\sigma, r)\phi(\pm i\sigma, s), & 0 < s \leq r \end{cases}.$$

These considerations suggest that, as in the classical theory, ϕ should give the distorted Fourier basis. Moreover, the limiting forms of the Whittaker functions [8, 13.14.20-1] show that ψ is the only decaying solution at ∞ i.e. it is the Weyl solution.

To proceed, we need to define the θ function to determine the spectral measure ρ . Due to the strong singularity of V at zero, we will not be able to pick the θ function as in (A.2). In [18], Gesztesy and Zinchenko proved that if V is real valued and H is in the limit point case at both end points then Weyl-m function exist provided that $Hf = zf$ has a solution $\tilde{\phi}(z, x)$ in O , an open neighborhood of \mathbb{R} , that is (a) analytic for $x \in (0, \infty)$ and $z \in O$ (b) real valued for $x \in (0, \infty)$ and $z \in \mathbb{R}$, and (c) in L^2 around $x = 0$ for $z \in \mathbb{C} \setminus \mathbb{R}$ with sufficiently small $|\Im z|$, see Hypothesis 3.1 of [18]. They further showed that if the singularity point of V and the end point of H agree, then Weyl-m function is a scalar function of z . We note, that $\phi(z, r)$ holds (a),(b) and (c).

We follow a similar argument that is used in [18], and find the fundamental system of solutions to $\mathcal{L}f = -z^2f$ at a reference point $x_0 = 1$. We take the first solution as $\phi(z, r)$ and pick a $\theta(z, r)$ such that $W(\phi(z, r), \theta(z, r)) = 1$, which we are free to do by Picard iteration from this point. Then, we must have

$$(A.6) \quad \psi(z, r) = \theta(z, r) + m(z)\phi(z, r), \quad m(z) = W(\theta(z, r), \psi(z, r)).$$

As in the proof of Proposition A.4, we use Stone's formula to obtain

$$(A.7) \quad [e^{it\mathcal{L}}f](r) = \int_0^\infty \int_0^\infty e^{it\sigma^2} \phi(i\sigma, r) \phi(i\sigma, s) f(s) \rho(d\sigma) ds, \quad \frac{d\rho}{d\sigma} = \frac{2\sigma}{\pi} \Im(m(i\sigma)).$$

Here, we have used that $\sigma(\mathcal{L}) = \sigma_{ac}(\mathcal{L}) = [0, \infty)$ since \mathcal{L} is a positive operator and for any $\sigma \in \mathbb{R}$ we have $\phi(\pm i\sigma, r), \psi(\pm i\sigma, r) \in L_\delta^2 \setminus L^2$ for $\delta > \frac{1}{2}$, where $L_\delta^2 := \{(1+r^2)^{-\frac{\delta}{2}}f \in L^2\}$. Note that $L_{-\delta}^2$, being the dual space of L_δ^2 , is dense in L^2 .

Finally, let us determine the density of ρ . Note that $\theta(z, r)$ has to be real analytic for $z = i\sigma$, therefore, we have to have $\theta(i\sigma, r) = \Re(\psi(i\sigma, r)) + b(\sigma)\phi(i\sigma, r)$ for some real valued $b(\sigma)$ as

$$W[\theta(i\sigma, \cdot), \phi(i\sigma, \cdot)] = \Re(W[\psi(i\sigma, \cdot), \phi(i\sigma, \cdot)]) = 1.$$

Hence, we compute

$$\begin{aligned} m(i\sigma) &= W(\theta(i\sigma, \cdot), \psi(i\sigma, \cdot)) \\ &= 2^{-1}W[\psi(i\sigma, \cdot) + \overline{\psi(i\sigma, \cdot)}, \psi(i\sigma, \cdot)] + b(\sigma) \\ &= \Im(\psi(i\sigma, \cdot)\overline{\psi'(i\sigma, \cdot)}) + b(\sigma). \end{aligned}$$

Moreover, we have as $r \rightarrow 0$,

$$\psi(i\sigma, r) = -1 + c(i\sigma)r - r \log r + O(r^{2-})$$

where $\Im(c(i\sigma)) = \sigma - q[\frac{\pi}{2} + \Im(\psi^{(0)}(1 - iq/(2\sigma)))]$ and $\psi^0(z)$ is digamma function. Therefore, $\Im(m(i\sigma)) = \Im(c(i\sigma))$ and using $\Im(\psi^0(1 + iy)) = -(2y)^{-1} + \frac{\pi}{2} \coth(\pi y)$, see [8, (5.7.5)], we obtain $d\rho(\sigma) = 2q\sigma[e^{\frac{q\pi}{\sigma}} - 1]^{-1}$ and this in (A.7) gives

$$(A.8) \quad \begin{aligned} [e^{it\mathcal{L}}f](r) &= -\frac{q}{2} \int_0^\infty \int_0^\infty e^{it\sigma^2} M_{\frac{iq}{2\sigma}, \frac{1}{2}}(2i\sigma r) M_{\frac{iq}{2\sigma}, \frac{1}{2}}(2i\sigma s) f(s) \sigma^{-1} [e^{\frac{q\pi}{\sigma}} - 1]^{-1} d\sigma ds \\ &= -\frac{q}{2} \int_0^\infty \int_0^\infty e^{itq^2\sigma^2} M_{\frac{i}{2\sigma}, \frac{1}{2}}(2iq\sigma r) M_{\frac{i}{2\sigma}, \frac{1}{2}}(2iq\sigma s) f(s) \sigma^{-1} [e^{\frac{\pi}{\sigma}} - 1]^{-1} d\sigma ds. \end{aligned}$$

Using the fact that $\mathcal{L} = rH_{0,q}r^{-1}$, we obtain (A.1).

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