DIRECTIONAL BALLISTIC TRANSPORT FOR PARTIALLY PERIODIC SCHRÖDINGER OPERATORS

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Abstract. We study the transport properties of Schrödinger operators on $\mathbb{R}^d$ and $\mathbb{Z}^d$ with potentials that are periodic in some directions and compactly supported in the others. Such systems are known to produce "surface states" that are confined near the support of the potential. We show that, under very mild assumptions, a class of surface states exhibits what we describe as directional ballistic transport, consisting of a strong form of ballistic transport in the periodic directions and its absence in the other directions. Furthermore, on $\mathbb{Z}^2$, we show that a dense set of surface states exhibit directional ballistic transport. In appendices, we generalize Simon's classic result on the absence of ballistic transport for pure point states [36], and prove a folklore theorem on ballistic transport for scattering states. In particular, this final result allows for a proof of ballistic transport for a dense set subset of $\ell^2(\mathbb{Z}^2)$ for periodic strip models.

1. Introduction

1.1. **Motivation and main results.** Recently, there has been much interest in Schrödinger operators with potentials supported near a hyperplane in $\mathbb{R}^d$, see, e.g., [2, 3, 14, 16, 20, 35] and the references therein. These models are physically natural because they describe a lower-dimensional system embedded in a higher-dimensional background and are expected to have applications in photonics [22, 27, 31]. Intuitively, one expects that any state should decompose into a piece that radiates into the background and one that is governed by lower-dimensional dynamics. That intuition was made rigorous in [3] where it was shown that if V is a real-valued bounded potential supported near a proper subspace of $\mathbb{R}^d$, then $L^2(\mathbb{R}^d)$ decomposes into the space of scattering states and the space of *surface states* given by

$$\mathcal{H}_{\text{sur}} = \{\psi \in L^2(\mathbb{R}^d) \mid \lim_{t\to\infty} \|\chi_{vt} e^{-it(H_0+V)}\psi\| = 0,\ \forall v > 0\},$$

where χ_{vt} is the indicator of the set of points of distance at least vt from the subspace. This result is agnostic to the exact choice of the potential V, so it is natural to ask how different properties of V affect the dynamics within $\mathcal{H}_{\text{sur}}$. In this paper, we are concerned with potentials that are periodic in the surface directions.

To be more precise, let us now specify the class of operators we wish to consider. Throughout, we will write $\mathbb{R}^d$ and $\mathbb{Z}^d$ as $\mathbb{R}^{n+m}$ and $\mathbb{Z}^{n+m}$, respectively, with the first n coordinates labeled by x and the last m labeled by y.

Definition 1.1. We say that a real-valued potential $V(x, y)$ in $L^\infty(\mathbb{R}^{n+m})$ or $\ell^\infty(\mathbb{Z}^{n+m})$ is *strip periodic* if V is compactly supported in the x variables and there exists m numbers $\{L_i\}_{i=1}^m$ in $\mathbb{R}$ (or $\mathbb{Z}$) such that for all $1 \le i \le m$

$$V(x, y + L_i\mathbf{e}_i) = V(x, y),\ \ \forall (x, y) \in \mathbb{R}^{n+m} \text{ (or } \mathbb{Z}^{n+m}).$$

where $\mathbf{e}_i$ is the standard basis.

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Let $\mathcal{H}$ be the Hilbert space $L^2(\mathbb{R}^{n+m})$ or $\ell^2(\mathbb{Z}^{n+m})$ depending on context and H_0 the free Hamiltonian, with convention specified below. For V strip periodic on $\mathcal{H}$, we consider the self-adjoint Schrödinger operator

$$H = H_0 + V. \tag{1.1.1}$$

Related classes of operators have been studied before by Davies-Simon [10] and more recently by Filonov and Filonov-Klopp [14, 16] as well as by Korotyaev-Saburova [27]. We review these works in Section 1.2.

One of the hallmarks of periodic Schrödinger operators is that a dense set of states exhibit *ballistic transport*. Let Q be the position operator on $\mathbb{R}^d$

$$Q\psi = \vec{q}\psi$$

where $\vec{q} = (q_1, \ldots, q_d)$, with domain

$$D(Q) = \left\{ \psi \in L^2(\mathbb{R}^d) : \int\limits_{\mathbb{R}^d} \|\vec{q}\|^2 |\psi(\vec{q})|^2 \, d\vec{q} < \infty \right\}.$$

For an operator A, we denote by $A_H(t)$ the Heisenberg-evolved operator $e^{itH}Ae^{-itH}$. We say that a state $\psi \in D(Q)$ undergoes ballistic transport if

$$\lim_{t\to\infty} \frac{1}{t} Q_H(t)\psi$$

exists and is non-zero. The fact that wave packets in crystals undergo ballistic transport is related to the motion of electrons in solids and has long been established mathematically [1]. Below, we recall various notions of ballistic transport that appear in the literature, but for now, we mention that the above is a particularly strong notion of transport.

With this in mind, one expects that for a strip periodic potential, surface states should enjoy ballistic transport *only in the periodic directions* while being strongly trapped in the transverse directions. This is because these states are confined to the vicinity of V and, therefore, should evolve mainly in accordance with its periodic structure. Our goal in this paper is to put this intuition on rigorous mathematical footing.

While this heuristic suggests that for $m = 1$ these systems are "essentially one-dimensional," we emphasize that the usual techniques used in the analysis of Schrödinger operators on the line or on a strip $\mathbb{Z}\times\{1, \cdots, L\}$ do not apply in any straightforward way. Indeed, the coupling between different vertical slices makes the problem multi-dimensional, and the unboundedness in the x direction creates the possibility of embedded eigenvalues. We further explain the difficulties inherent to these models below.

To state our results, on $\mathbb{R}^{n+m}$ or $\mathbb{Z}^{n+m}$ let X and Y be the partial position operators

$$X\psi(x, y) = (x_1, \ldots, x_n)\psi(x, y), \; Y\psi(x, y) = (y_1, \ldots, y_m)\psi(x, y),$$

with the natural domains, so that $Q = (X, Y)$. We make the following definition:

Definition 1.2. We say that $\psi \in D(Q)$ exhibits *directional ballistic transport* if

$$\lim_{t\to\infty} \frac{1}{t} X_H(t)\psi = 0$$

and

$$\lim_{t\to\infty} \frac{1}{t} Y_H(t)\psi$$

exists and is non-zero.

Clearly a state that exhibits directional ballistic transport also exhibits ballistic transport in the sense described above.

Our most satisfactory result is on $\mathbb{Z}^{1+1}$:

Theorem 1.3. *Let V be a strip periodic potential on $\mathbb{Z}^{1+1}$ and H the associated Schrödinger operator $H = H_0 + V$. Then a dense subset of $\mathcal{H}_{\mathrm{sur}}$ exhibits directional ballistic transport.*

Remark 1.4. We believe that our proof method should extend to $\mathbb{Z}^{1+m}$. However, as an effective demonstration of our ideas, we treat the simplest case of $1 + 1$ dimensions.

We note that in the strip periodic setting, $\mathcal{H}_{\mathrm{sur}}$ has a concrete description in terms of the Floquet theory of H; see Section 1.2. Basically, one writes H as a direct integral of operators $H(k)$, and the surface states emerge as the integral of pure point states on each fiber. Since the orthogonal complement of the space of surface states consists of scattering states, one may obtain a more complete transport statement by showing that, in this setting, a dense subset of the scattering states undergoes ballistic transport. We prove a general result to this effect in Appendix B, which in particular yields

Corollary 1.5. *Let V be a strip periodic potential on $\mathbb{Z}^{1+1}$ and let H be the associated Schrödinger operator $H = H_0 + V$. Then a dense subset of $\ell^2(\mathbb{Z}^{1+1})$ exhibits ballistic transport.*

As explained below, the main difficulty in establishing directional ballistic transport is the possibility of point spectrum embedded within the essential spectrum on a fiber. In continuum and higher dimensional settings, this difficulty is particularly acute, and we obtain directional ballistic transport only for a class of *unembedded surface states*, $\mathring{\mathcal{H}}_{\mathrm{sur}} \subset \mathcal{H}_{\mathrm{sur}}$, defined below. In this generality, we prove:

Theorem 1.6. *Let V be a strip periodic potential on $\mathbb{R}^{n+1}$ or $\mathbb{Z}^{n+1}$ and H the associated Schrödinger operator $H = H_0 + V$. Then a dense subset of states in $\mathring{\mathcal{H}}_{\mathrm{sur}}$ exhibits directional ballistic transport.*

Remark 1.7. The proof of Theorem 1.6 establishes transport for states that are composed of eigenfunctions of $H(k)$ with eigenvalues that vary sufficiently smoothly in k. When there is only one periodic direction, standard perturbation theory guarantees that the unembedded eigenvalues and eigenprojectors may be parameterized analytically away from a discrete set. In higher dimensions, establishing joint analyticity in several variables is more delicate; see the remarks in [40]. Thus, it seems possible to extend our results to $m > 1$ by adapting the approach of [40] or [19], but we do not pursue this here.

At this point, it is worth mentioning that the above theorems may be vacuous if the surface subspace is empty. To this end, we give a natural sufficient condition in 1+1 dimensions (Proposition 2.10) for the existence of states in $\mathring{\mathcal{H}}_{\mathrm{sur}}(H)$. As is typical when showing the existence of bound states, this involves imposing some sort of negativity assumption on V to force the existence of spectrum below 0.

Before turning to a more detailed discussion, we also mention some auxiliary results that may be of independent interest. First, the proof of the absence of ballistic transport in the x-direction is closely related to Simon's classic result on the absence of ballistic transport for pure point states [36]. To our knowledge, the forms of this theorem that appear in the literature pertain to operators with only pure point spectrum. In contrast, for our purposes, we must generalize to operators that may also have continuous spectrum (see Appendix A). This result is natural, in particular, because many important operators have both spectral types, especially on $\mathbb{R}^d$. In fact, the existence of bounded potentials in the continuum setting that induce completely pure point spectrum is a hard problem. In one dimension, this was established for the continuum Anderson model with absolutely continuous random variables by Kotani-Simon [28] and for singular random variables by Damanik-Sims-Stolz [8]. In higher dimensions, this remains a major open problem.

Second, Corollary 1.5 is a consequence of the fact that scattering states exhibit ballistic transport - at least in our setting. While this fact has been suggested in the literature [24, 32] for short-range potentials, we are not aware of a recorded proof so we supply one in Appendix B. In particular, we provide a general criterion for a scattering state to exhibit ballistic transport *in a certain direction.* This is inspired by Cook's criterion for the existence of the wave operator.

1.2. **Context: partially periodic operators and surface states.** As mentioned above, operators that are periodic only in some coordinate directions have been studied at least since the work of Davies and Simon [10]. Like fully periodic operators, they may be studied via the (partial) Floquet transform. We explain its properties in Proposition 2.2, but for now it suffices to know that it is a unitary transformation that conjugates a partially periodic operator H in $n+m$ dimensions to the direct integral of operators

$$\int_{\mathbb{T}^*}^{\oplus} H(k)\, \frac{dk}{|\mathbb{T}^*|}$$

where $\mathbb{T}^*$ is an m-dimensional torus and each $H(k)$ is a self-adjoint operator on the cylinder $\mathbb{R}^n \times \mathbb{T}^*$. By Proposition 2.4 below, the surface subspace is equal to

$$\int_{\mathbb{T}^*}^{\oplus} \mathcal{H}_{\mathrm{pp}}(H(k))\, \frac{dk}{|\mathbb{T}^*|}.$$

On the orthogonal complement of this subspace, H has purely absolutely continuous spectrum, as shown in Proposition 2.5. Thus, the spectral theory of H is intimately tied to the variation in k of $\mathcal{H}_{\mathrm{pp}}(H(k))$.

In a remarkable pair of papers, Filonov-Klopp [16] and Filonov [14] have shown that if V is periodic in y, under different decay assumptions in the x-directions, the operator H has either no eigenvalues or purely absolutely continuous spectrum. Specifically, if $|V(x,y)| < C|x|^{-\rho}$ for $\rho > 1$ then H has no eigenvalues [14] and if V decays superexponentially then H has purely absolutely continuous spectrum [16]. Prior to Filonov [14], Hoang-Radosz [20] obtained a similar result on the absence of eigenvalues for Helmholtz and Schrödinger operators on $\mathbb{R}^2$. We also mention the work of Korotyaev-Saburova [27], who studied analogs of strip periodic potentials on more general graphs. For these models, they obtained certain estimates on the locations of the bands.

The main obstacle in establishing these results is the possibility of eigenvalues embedded in the essential spectrum of the fibered operator $H(k)$. Indeed, one needs to understand the variation of these eigenvalues in k, and the perturbation theory of embedded eigenvalues is typically quite challenging and must be treated in a context-dependent fashion. The strategy adopted by the aforementioned authors is to analytically continue the resolvent on some weighted space from the upper half-plane up to or across the real axis and then to study how this operator behaves as k changes. Per our understanding, these results do not show that the eigenvalues are analytic functions of k due to the possibility of resonances, i.e., poles of the resolvent that are not eigenvalues. The transport properties of a periodic system were established in [1] using that the energies are differentiable and nonconstant in the quasimomentum, k, almost everywhere. Thus, we require stronger control on the variation of these embedded energies than has previously been obtained. We show that on $\mathbb{Z}^{1+1}$ this may be accomplished by introducing a new fiber-wise transfer matrix formalism that in some sense allows us to study the pure point part of the resolvent directly, see Section 4.

1.3. **Context: ballistic transport.** Ballistic transport is well-studied mathematically and remains an active area of research. In the 1990s, Asch-Knauf [1] demonstrated ballistic transport in the strong sense introduced above for periodic potentials and a dense set of initial states. This was later extended to the setting of periodic Jacobi matrices in one and arbitrary dimensions, respectively in [7, 13]. This strong form of ballistic transport is only known for certain classes of operators; in addition to the aforementioned periodic results, results are known for certain limit-periodic operators on $\mathbb{Z}$ [12], and on $\mathbb{R}$ [41]. In many other settings, the accessible notions of transport are strictly weaker. We refer the reader to [9] for further background on this and other notions of transport, as well as further works in the one-dimensional, almost periodic setting [18, 23, 42, 43, 44].

Besides the aforementioned work of Asch-Knauf and the result of Fillman [13], little is known in multi-dimensional settings, even for weaker forms of ballistic transport. Two results in that vein are the work of Karpeshina et al. [24, 25] where ballistic lower bounds for the Abel mean of the position operator are shown for certain quasi-periodic and limit-periodic operators on $\mathbb{R}^2$, and for generic quasi-periodic operators on $\mathbb{R}^d$, for $d \geq 2$ respectively. Thus, our work contributes to the understanding of transport beyond one spatial dimension.

1.4. **Overview of the proofs.** As mentioned above, the main obstacle in establishing ballistic transport in this context is the possibility of embedded surface states - see below for a precise definition. Therefore, this study is divided into two parts: the first (Section 2) deals with the unembedded surface states in general either in the discrete or continuum setting, with arbitrarily many decaying dimensions, and one periodic direction, while the second (Section 4) studies all surface states in the special case of $\mathbb{Z}^{1+1}$.

In the first case, we are able to use the perturbation theory of isolated eigenvalues to establish ballistic transport for these unembedded surface states in great generality. As mentioned above, this restricts our result to one periodic direction. Furthermore, we produce a sufficient condition for the existence of such states. While the perturbation theory used in this section is standard, the presence of essential spectrum incurs some difficulties in the proof of directional ballistic transport that are addressed in Theorem 2.15 and Proposition 2.17.

The study of the variation of embedded eigenvalues is much more delicate. For this, we emulate the strategy outlined by Wilcox [40]: find an analytic function $F(E,k)$ whose zero set coincides with the eigenvalues of $H(k)$ and then appeal to the theory of real analytic varieties to show that the eigenvalues are suitably smooth. Unlike in the classical setting of a fully periodic potential, we encounter the difficulty that our resolvent is not compact, so $F(E,k)$ may not be constructed as a Fredholm determinant. To use one-dimensional tools, we specialize to $\mathbb{Z}^2$. The advantage is that the Floquet transform reduces the system to an analytic family of L-many coupled discrete Schrödinger operators, each with a compactly supported potential. Thus, we may examine the eigenvalue problem via transfer matrices. For any fixed k, as $x \to \pm\infty$, the space of decaying solutions is finite-dimensional, so we can reduce the existence of an eigenvalue of $H(k)$ to an analytically varying connection problem between these two subspaces across the support of V. By forming the determinant associated with this connection problem, we are able to construct the "partial Bloch variety" of H. The theory of analytic varieties allows us to conclude that each eigenvalue may be taken to be analytic almost everywhere (as a function of k). This analysis is more complicated than the perturbation theory of matrices because the variables E and k enter this determinant non-linearly. For this purpose, Wilcox [40] used deep results of Cartan [4] and Whitney-Bruhat [39] on the structure of real analytic varieties, but to keep our work self-contained we develop the necessary machinery ourselves, see in particular, Lemma 4.4 and Appendix C. As mentioned above, it seems to us that this methodology should extend to $\mathbb{Z}^{1+m}$, the crucial element being the ability to use transfer matrices on each fiber. On the other hand, it would be interesting to see if this approach could be adapted to a continuum setting or to accommodate more decaying directions.

1.5. **Outline of the paper.** This paper is arranged as follows:

Section 2 proves the existence of the unembedded surface states and the analytic variation with the quasi momenta, culminating in Theorem 2.15. In Section 2.1, we introduce the Floquet transform in the continuum setting and present some immediate results from it. In Section 2.2, we find sufficient conditions on the Floquet transform of a state ensuring membership in $D(X)$ and $D(Y)$. In Section 2.3, we establish a sufficient condition for unembedded surface states existence, with certain dimension restrictions, and prove that these will vary analytically with k, which allows us to prove Theorem 2.15.

Section 3 discusses some scattering results that allow us to prove the existence of a dense subset of the scattering states that exhibit ballistic transport. This, in combination with Theorem 2.15, allows us to prove Theorem 1.6.

In Section 4, we specialize to $\mathbb{Z}^{1+1}$ to study embedded surface states. In Section 4.1, we start by introducing the setting and relevant notation. In Section 4.2, we study the spectral theory of the free Hamiltonian on a fiber. Then, in Section 4.3, we reformulate the problem and show that the eigenvalues are encoded as the zero set of a determinant of a finite-dimensional matrix. Finally, in Section 4.4, we establish that the eigenvalues vary locally analytically, which allows us to prove Theorem 1.3.

Finally, in Appendices A and B, we establish some results that might be of interest outside the scope of this work. The first concerns the absence of ballistic transport for pure point states for operators that may have continuous spectrum, and the second is a sufficient condition for asymptotically free states to exhibit ballistic transport. In Appendix C, we provide a form of the Weierstrass preparation theorem for real analytic functions, which is then used to prove the analytic variation of the embedded eigenvalues.

In Appendix D we supply a glossary for the different notations and conventions used throughout.

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2. Continuum strip periodic operators: existence and analyticity of unembedded surface states

2.1. Setting and Floquet theory.

In this section, we consider the operator

$$Hu = -\Delta u + Vu$$

acting on $L^2(\mathbb{R}^{n+m})$ with domain $H^2(\mathbb{R}^{n+m})$ where $-\Delta$ is the negative Laplacian and V is strip periodic with x-support within the ball of radius R and periods $\{L_i\}_{i=1}^m$. We note that since $V \in L^\infty(\mathbb{R}^{n+m})$, H is self-adjoint as a consequence of the Kato-Rellich theorem.

Remark 2.1. The results of this section may easily be extended to strip periodic Schrödinger operators on $\mathbb{Z}^{n+m}$. We mention that all results are independent of dimension except Lemma 2.10, where the necessary assumptions on the potential V will change according to the dimension n. In particular, for $n \geq 3$, we will need the average of V to be smaller than some absolute negative constant, rather than it simply being negative (see the remark before Theorem 5 in [6] for the discrete setting, and for the continuous setting see [21]). In particular, all statements leading to the proof of Theorem 1.6 apply in full generality, but the content of the theorem may be vacuous without further dimension-dependent assumptions on V.

For the Floquet theory of the operator H, let $W = \mathbb{R}^n \times \prod_{j=1}^m [0, L_j)$ and let

$$\tilde{H}^2 = \{f \in H^2(W) \mid f\mid_{y=0} = f\mid_{y=L}, \frac{\partial}{\partial y} f\mid_{y=0} = \frac{\partial}{\partial y} f\mid_{y=L}\}$$

Define for $f \in L^2(\mathbb{R}^{n+m})$ the *partial Floquet transform*

$$(Uf)(x, y, k) = \sum_{n \in \mathbb{Z}^m} e^{-i\langle k, y + \sum_{j=1}^m n_j L_j \rangle} f(x, y + \sum_{j=1}^m n_j L_j)$$

for $(x,y) \in \mathbb{R}^{n+m}$ and $k \in \mathbb{T}^* = \mathbb{R}^m/\Gamma^*$ for Γ^* the lattice dual to the lattice of periodicity

$$\{z \in \mathbb{R}^m \mid \langle z, (L_1 n_1, \cdots, L_m n_m)\rangle \in 2\pi\mathbb{Z}\, \forall n \in \mathbb{Z}^m\}.$$

As an $L^2(\mathbb{T}^*) \otimes L^2(W)$ convergent sum, the Floquet transform defines a bounded map from $L^2(\mathbb{R}^{n+m})$ to $L^2(\mathbb{T}^*) \otimes L^2(W)$. The following properties of the Floquet transform are standard, see, for instance, Section 4 of [29]:

Proposition 2.2. *The map $f \mapsto Uf$ has the following properties:*

(1) U is a unitary map from $L^2(\mathbb{R}^{n+m})$ to $L^2(W) \otimes L^2(\mathbb{T}^*)$.

(2) We have the unitary equivalence

$$UHU^* = \int_{\mathbb{T}^*}^{\oplus} H(k)\, \frac{dk}{|\mathbb{T}^*|},$$

where

$$H(k) = -\left(\nabla + i\begin{pmatrix}0\\k\end{pmatrix}\right)^2 + V = -\Delta_x - (\nabla_y + ik)^2 + V = H_0(k) + V$$

self-adjoint on $\tilde{H}^2$.

The reader may find the necessary background on direct integrals of Hilbert spaces in [33]. For the discrete setting, we will also define a Floquet transform with similar properties; see Section 4.1.

We start by noting that:

Proposition 2.3. *The spectrum of $H_0(k)$ is absolutely continuous and is given by $[\|k\|^2, \infty)$. Furthermore, V is relatively compact to $H_0(k)$ so that $\sigma_{\text{ess}}(H(k)) = \sigma(H_0(k))$.*

Proof. The first claim comes from regarding $L^2(\mathbb{R}^{n+m})$ as $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^m)$ and using the Floquet transform, to write

$$H_0(k) = (-\Delta_x) \otimes \mathrm{Id} - \mathrm{Id} \otimes (\nabla_y + ik)^2$$

The spectral measure of operators of the form $A \otimes \mathrm{Id} + B \otimes \mathrm{Id}$ is the convolution of the spectral measures of A and B [17], each of which is absolutely continuous in this case.

The second claim follows immediately from Lemma 5.1 of [15], which shows that V is relatively compact to $H_0(k)$. $\square$

Now recall that the surface subspace is defined as

$$\mathcal{H}_{\text{sur}} = \{\psi \in L^2(\mathbb{R}^{n+m}) \mid \lim_{t\to\infty} \|\chi_{vt} e^{-itH}\psi\| = 0,\ \forall v > 0\}, \tag{2.1.1}$$

where for $R > 0$, χ_R is the indicator function of the set $\{|x| > R\}$. In the strip periodic setting, we have the following description of $\mathcal{H}_{\text{sur}}$ that we will use henceforth:

Proposition 2.4. *For H strip periodic, we have that*

$$\mathcal{H}_{\text{sur}} = \int_{\mathbb{T}^*}^{\oplus} \mathcal{H}_{\text{pp}}(k)\, \frac{dk}{|\mathbb{T}^*|}.$$

We postpone the proof of this Proposition to Section 3 where the necessary scattering theory is developed. For now, we prove a more basic scattering statement. First, define the *wave operator* as the strong limit

$$\Omega = \operatorname*{s-lim}_{t\to\infty} e^{itH} e^{-itH_0}.$$

The range of the wave operator consists of the *scattering states* which are those whose evolution is close, in $\mathcal{H}$, to the free evolution as $t \to \infty$. In fact, since free waves naturally exhibit ballistic transport, one can show that this is true for the scattering states as well under certain assumptions on V; see Appendix B for the details.

In this context, we have the following asymptotic completeness result. Recall that we denote by $\mathcal{H}$ either $\ell^2(\mathbb{Z}^d)$ or $L^2(\mathbb{R}^d)$. Then:

Proposition 2.5. *For H strip periodic, we have that $\Omega\psi$ exists for all $\psi \in \mathcal{H}$. In addition,*

$$\mathcal{H} = \int_{\mathbb{T}^*} \mathcal{H}_{\mathrm{pp}}(k)\, dk \oplus \mathrm{Ran}(\Omega),$$

where H has purely absolutely continuous spectrum on $\mathrm{Ran}\,\Omega$.

Proof. For each $k \in \mathbb{T}^*$, define the wave operator

$$\Omega(k) = \operatorname*{s-lim}_{t\to\infty} e^{itH(k)} e^{-itH_0(k)}$$

on $L^2(W)$. In the continuum setting, Theorem 1.1(c) of [15] shows that the strong limit defining $\Omega(k)$ exists, and the operator is complete in the sense that

$$L^2(W) = \mathcal{H}_{\mathrm{pp}}(k) \oplus \mathrm{Ran}\,\Omega(k).$$

In the discrete setting, the difference between $H(k)$ and $H_0(k)$ is finite rank so the existence and completeness of the wave operators is an immediate consequence of the Kato-Rosenblum Theorem [34, Thm XI.8], which requires the difference to be only trace class.

Now, from the proof of Theorem 6.3 of [3], we see that

$$\Omega = \int_{\mathbb{T}^*}^{\oplus} \Omega(k)\, \frac{dk}{|\mathbb{T}^*|}.$$

Note that the result in that work is stated for $\mathbb{R}^{1+m}$, but this part of the proof generalizes immediately to continuum and discrete settings of arbitrary dimension. Integrating, we have that

$$\mathcal{H} = \int_{\mathbb{T}^*}^{\oplus} \mathcal{H}_{\mathrm{pp}}(k)\, \frac{dk}{|\mathbb{T}^*|} \oplus \mathrm{Ran}\,\Omega$$

as desired. □

2.2. **Domain considerations.**

Recall the directional position operators

$$X\psi = x\psi(x,y), \;\; Y\psi = y\psi(x,y)$$

with corresponding domains

$$D(X) = \{\psi \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \|x\|^2 |\psi(x,y)|^2\, dydx < \infty\},$$
$$D(Y) = \{\psi \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \|y\|^2 |\psi(x,y)|^2\, dydx < \infty\}.$$

Membership in $D(Y)$ may be verified on the Floquet side via the following Paley-Wiener type theorem (see also Theorem 4.2 in [29]):

Lemma 2.6. *Suppose that $\varphi \in L^2(\mathbb{R}^d)$ is such that $U\varphi(k,\cdot,\cdot)$ is C^2 w.r.t. k as a function taking values in $L^2(\mathbb{R}^n \times \mathbb{T}^*)$. Then $\varphi \in D(Y)$.*

Proof. We give the proof for $L_i = 1$ for all i, as the adaptation to more general periods is straightforward. Using the differentiability of $U\varphi$, we may integrate by parts twice with respect to k_j in the inversion formula

$$\varphi(x, y + \gamma) = \int_{\mathbb{T}^*} e^{ik\cdot(y+\gamma)} U\varphi(k, x, y)\, \frac{dk}{|\mathbb{T}^*|}$$

for $y \in [0, 1)^m$ and $\gamma \in \mathbb{Z}^m$, to find that

$$\int_{\gamma_j}^{\gamma_j+1} \int_{\mathbb{R}^n} |\varphi(x, y)|^2\, dx dy_j \leq \frac{C}{\gamma_j^4 + 1}$$

for a constant $C > 0$ that is independent of γ. Thus,

$$\begin{aligned}\int_{\mathbb{R}^d} \|y\|^2 |\varphi(x, y)|^2\, dy dx &= \sum_{\gamma\in\mathbb{Z}^m} \int_{\gamma_1}^{\gamma_1+1} \cdots \int_{\gamma_m}^{\gamma_m+1} \int_{\mathbb{R}^n} \|y\|^2 |\varphi(x, y)|^2\, dx dy \\ &\leq C \sum_{\gamma\in\mathbb{Z}^m} \frac{(\|\gamma\| + 1)^{2m}}{\prod_{j=1}^m (\gamma_j^4 + 1)} < \infty\end{aligned}$$

as is required. □

Membership in $D(X)$ is a little more subtle. First, we need the following lemma.

Lemma 2.7. *Let $g \in L^2(\mathbb{R}^n)$ be compactly supported and let $E > 0$. Then any $f \in H^2(\mathbb{R}^n)$ solving the inhomogenous Schrödinger equation*

$$-\Delta f + g = Ef \tag{2.2.1}$$

is identically 0.

Proof. For $n = 1$, Equation 2.2.1, for $|x| > R$, will become

$$-\frac{d^2}{dx^2} f = Ef,$$

which has no L^2 solution other than $f \equiv 0$. Thus, we assume that $n > 1$. It suffices to show that if $Y(\omega)$ is any eigenfunction of the spherical Laplacian, then $\langle f, Y\rangle_{L^2(\mathbb{S}^{n-1})} \equiv 0$. Let $Y(\omega)$ be such an eigenfunction of eigenvalue $c = -\ell(\ell + n - 2)$, for some $\ell \in \mathbb{N}_0$. By integrating (2.2.1) against $\overline{Y}(\omega)$ (in the angular variables), and expressing Δ in spherical coordinates, we obtain the ODE

$$-r^{1-n} \frac{d}{dr}\left(r^{n-1} \tilde{f}'(r)\right) - \frac{c}{r^2} \tilde{f}(r) + \tilde{g}(r) = E\tilde{f}(r),$$

where

$$\tilde{f}(r) = \int_{\mathbb{S}^{n-1}} f(r\omega)\overline{Y}(\omega)\, d\omega$$

$$\tilde{g}(r) = \int_{\mathbb{S}^{n-1}} g(r\omega)\overline{Y}(\omega)\, d\omega.$$

A priori this ODE must be interpreted weakly, but since weak and strong solutions of ODEs coincide, we may consider $\tilde{f}$ to be twice-differentiable. In particular, we study the ODE outside of the support of g, when $r > R$, say. For $E > 0$, we make the substitution $h(r) = r^{\frac{n}{2}-1} f(r/\sqrt{E})$ to obtain

$$r^2 h''(r) + r h'(r) + (r^2 - [c - (1 - \frac{n}{2})^2])h(r) = 0,$$

which is Bessel's equation. We therefore obtain the representation for $\tilde{f}(r)$, $r > R$

$$\tilde{f}(r) = r^{1-\frac{n}{2}}(AJ_\nu(\sqrt{E}r) + BY_\nu(\sqrt{E}r)),$$

where J_ν and Y_ν are the Bessel functions of the first and second kind, respectively, of order $\nu = \sqrt{c - (1-\frac{n}{2})^2}$.

Using the well-known asymptotics (10.17.3-4) of [11], we see that for $r > R$

$$\tilde{f}(r) = \left(\frac{2}{\pi\sqrt{E}}\right)^{\frac{1}{2}} r^{\frac{1-n}{2}} \left[A\cos(\sqrt{E}r - \frac{1}{2}\nu\pi - \frac{1}{4}\pi) + B\sin(\sqrt{E}r - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)\right] + O(r^{-\frac{1+n}{2}}).$$

Since the $O(r^{-\frac{1+n}{2}})$ term is $L^2(\mathbb{R}^n)$, we see that that the first term can only be $L^2(\mathbb{R}^n)$ if $A = B = 0$. By existence and uniqueness of ODEs, we conclude that $\tilde{f} \equiv 0$. Because $Y(\omega)$ was arbitrary, we see that $f \equiv 0$.

□

With this in hand, we obtain the following fiber-wise result:

Proposition 2.8. *Let ϕ be an eigenfunction of $H(k)$ of eigenvalue E, such that*

$$\forall j \in \mathbb{Z}^m, E \neq \left\|k + \frac{2\pi}{L}j\right\|^2,$$

and let

$$\delta = \min_{j\in\mathbb{Z}^m}\left\{\left\|k + \frac{2\pi}{L}j\right\|^2 - E \mid \left\|k + \frac{2\pi}{L}j\right\|^2 - E > 0\right\}.$$

which is positive by definition, and we used the shorthand $\frac{j}{L}$ to denote the vector in $\mathbb{R}^m$ with elements $\frac{j_i}{L_i}$. Then $\phi \in D(X)$ and

$$\|X\phi\| < C\max\left\{\frac{1}{\delta^{\frac{1}{2}}}, \frac{1}{\delta^{\frac{3}{2}}}\right\}\|\phi\|,$$

where $C = C(n, R, \|V\|_\infty)$.

Proof. By homogeneity, we assume that ϕ is L^2-normalized. For $j \in \mathbb{Z}^m$, let

$$\xi_j(y) = \frac{e^{i2\pi\sum_{u=1}^m \frac{j_u y_u}{L_u}}}{\sqrt{\prod_{u=1}^m L_u}}$$

and

$$\phi_j(x) = \int\limits_{\prod_{u=1}^m [0,L_u)} \phi(x,y)\overline{\xi}_j(y)\,dy$$

so that

$$\phi(x,y) = \sum_{j\in\mathbb{Z}^m}\phi_j(x)\xi_j(y),\ \|\phi\|_2^2 = \sum_{j\in\mathbb{Z}^m}\|\phi_j\|_2^2. \tag{2.2.2}$$

Then integrating the equation $H(k)\phi = E\phi$ in y against $\overline{\xi}_j(y)$ shows that

$$-\Delta_x\phi_j(x) + \|k + 2\pi\frac{j}{L}\|^2\phi_n(x) + (V\phi)_j(x) = E\phi_n(x),$$

where

$$(V\phi)_j(x) = \int\limits_{\prod_{u=1}^m [0,L_u)} V(x,y)\phi(x,y)\overline{\xi}_j(y)\,dy.$$

Denote $\alpha_j = \sqrt{\|k + \frac{2\pi}{L} j\|^2 - E}$. So we can write

$$-\Delta_x \phi_j(x) + \alpha_j^2 \phi_j(x) = -(V\phi)_j(x).$$

Treating the compactly supported $(V\phi)_j(x)$ as an inhomogeneous term, we may take $\alpha_j^2 > 0$ by Lemma 2.7. Thus, applying the free resolvent, $R_0(-\alpha_j^2)$, we find the implicit equation for ϕ_j:

$$\phi_j(x) = -[R_0(-\alpha_j^2)(V\phi)_j](x).$$

This allows for the estimate

$$\begin{aligned} \|X\phi_j\| = \|\mathcal{F}(X[R_0(-\alpha_j^2)(V\phi)_j])\| &= \left\| \nabla_\xi \left(\frac{1}{\|\xi\|^2 + \alpha_n^2} (\mathcal{F}(V\phi)_j)(\xi) \right) \right\| \\ &\leq 2 \left\| \Xi \frac{1}{(\|\xi\|^2 + \alpha_n^2)^2} (\mathcal{F}(V\phi)_j)(\xi) \right\| + \left\| \frac{1}{\|\xi\|^2 + \alpha_n^2} \nabla_\xi (\mathcal{F}(V\phi)_j)(\xi) \right\| \end{aligned}$$

for Ξ the position operator in the ξ variable. From the above, it follows that

$$\begin{aligned} \|X\phi_j\| &\leq \frac{3\sqrt{3n}}{8} \frac{1}{\delta^{\frac{3}{2}}} \|(V\phi)_j\| + \frac{1}{\delta^{\frac{1}{2}}} \|X(V\phi)_j\| \\ &\leq \left(\frac{3\sqrt{3n}}{8} + R \right) \max\left\{ \frac{1}{\delta^{\frac{1}{2}}}, \frac{1}{\delta^{\frac{3}{2}}} \right\} \|(V\phi)_j\|. \end{aligned}$$

Squaring and summing over j yields the result. □

2.3. **Unembedded surface states: existence and directional ballistic transport.** Recall that the surface subspace is given by

$$\mathcal{H}_{\text{sur}} = \int_{\mathbb{T}^*}^{\oplus} \mathcal{H}_{\text{pp}}(k) \, \frac{dk}{|\mathbb{T}^*|}.$$

We note that by Appendix 2 of [10], the map $k \mapsto \mathcal{P}_{\text{pp}}(k)$ is a measurable function of k, in the sense that $\langle \phi, \mathcal{P}_{\text{pp}}(k)\psi \rangle$ is measurable for all $\phi, \psi \in L^2(W)$, and therefore the above integral is well-defined. We shall frequently integrate over operator-valued functions of k, but we will not comment further on their measurability as it will follow in every instance from the considerations in [10].

The set of unembedded surface states $\mathring{\mathcal{H}}_{\text{sur}}$ is the set of surface states that on each fiber are supported away from the essential spectrum of $H(k)$ and the thresholds $\|k + \frac{2\pi j}{L}\|^2$ for $j \in \mathbb{Z}^m$. Namely, let

$$\mathring{\sigma}(k) = (-\infty, \|k\|^2) \setminus \left\{ \left\| \frac{2\pi j}{L} + k \right\|^2 \mid j \in \mathbb{Z}^m \right\}$$

so that

$$\mathring{\mathcal{H}}_{\text{sur}} = \int_{\mathbb{T}^*}^{\oplus} \chi_{\mathring{\sigma}(k)}(H(k)) \, \frac{dk}{|\mathbb{T}^*|},$$

where $\chi_{\mathring{\sigma}(k)}$ is the indicator function of this set.

Remark 2.9. The excision of the thresholds is natural in view of their pathological role in the analysis of [16]. Nonetheless, they only make an appearance in our argument insofar as eigenvectors of $H(k)$ at one of these thresholds may fail to lie in $D(X)$ when $n = 5, 6$. Otherwise, we may use the definition $\mathring{\mathcal{H}}_{\text{sur}} = \int_{\mathbb{T}^*}^{\oplus} \chi_{(-\infty, \|k\|^2)}(H(k)) \, \frac{dk}{|\mathbb{T}^*|}$.

Now, we show that this subspace is not empty under some mild conditions, at least in low dimensions:

Lemma 2.10. *Let $n = m = 1$. There is some absolute constant $C < 0$, such that if $\int_W V(x, y)\, dx\, dy < C$, then for each $k \in \mathbb{T}^*$, $H(k)$ has an eigenvalue in $\mathring{\sigma}(k)$.*

Proof. Let $\varphi_n(x, y) = (nL_1)^{-\frac{1}{2}}\varphi(x/n)$ for $\varphi \in C_c^\infty$ that is 1 for $x \in [-R, R]$. Regarding φ_n as an L^2 function on W, we compute that

$$\begin{aligned}\langle \varphi_n, H(k)\varphi_n\rangle &= \langle \varphi_n, -\Delta_x\varphi_n\rangle + \langle \varphi_n, -(\nabla_y + ik)^2\varphi_n\rangle + \langle \varphi_n, V\varphi_n\rangle = \\ &= \|\varphi_n'\|^2 + k^2\|\varphi_n\|^2 + \int_W V(x, y)|\varphi_n(x)|^2\, dx\, dy \\ &= \|\varphi_n'\|^2 + k^2\|\varphi_n\|^2 + \int_W V(x, y)\, dx\, dy\end{aligned}$$

We note that $\|\varphi_n\| = \|\varphi\|$, and that we have

$$\|\varphi_n'\|^2 = \int_{\mathbb{R}} \frac{1}{n^3}|\varphi'(\frac{x}{n})|^2\, dx = \frac{1}{n^2}\|\varphi'\|^2.$$

Thus, we obtain

$$\frac{\langle \varphi_n, H(k)\varphi_n\rangle}{\|\varphi_n\|^2} = n^{-2}\frac{\|\varphi'\|^2}{\|\varphi\|^2} + k^2 + \|\varphi\|^{-2}\int_W V(x, y)\, dx\, dy.$$

By choosing n large enough and C sufficiently negative, this last expression may be made smaller than $\|k^2 + \frac{2\pi j}{L_1}\|^2$ for any $j \in \mathbb{Z}$. This shows that $H(k)$ has spectrum below all of the thresholds, which must therefore be discrete spectrum per Proposition 2.3. ☐

Now, let $0 \le N(k) \le \infty$ be the number of eigenvalues of $H(k)$ below $\|k\|^2$. For any $k \in \mathbb{T}^*$ and $n \in \mathbb{N}$, let $\mathring{\pi}_n(k)$ be the eigenprojector associated to the nth eigenvalue of $H(k)$ below $\|k\|^2$ if $n \le N(k)$ and 0 otherwise. Also let $E_n(k)$ be the eigenvalue associated to $\mathring{\pi}_n(k)$ and let $S_n = \{k \in \mathbb{T}^* \mid \mathring{\pi}_n(k) \neq 0\}$. Note that by construction $E_n(k) \neq E_m(k)$ when $n \neq m$, as at a crossing the rank of $\mathring{\pi}_n(k)$ jumps.

We will show that a subset of $\mathring{\mathcal{H}}_{\text{sur}}$, denoted $\mathring{\mathcal{C}}$, exhibits ballistic transport in the y-directions. When $m = 1$, we will be able to show that it is in fact dense in $\mathring{\mathcal{H}}_{\text{sur}}$. It is given by:

$$\mathring{\mathcal{C}} := \bigcup_{\ell=1}^{\infty}\left\{\psi \in \mathring{\mathcal{H}}_{\text{sur}} \mid U\psi = \sum_{n=1}^{\ell}\mathring{\pi}_n(k)U\psi \text{ and for all } 1 \le n \le \ell,\ \mathring{\pi}_n(k)U\psi \in C^\infty(S_n)\right\}.$$

Here, as in [29], that $\mathring{\pi}_n(k)U\psi$ is C^∞ at a point means that it is smooth as a mapping valued in $L^2(W)$.

Remark 2.11. The definition of $\mathring{\pi}_n(k)$ is somewhat inconvenient because, as written, the rank of $\mathring{\pi}_n$ will jump at eigenvalue crossings below the eigenvalue corresponding to $\mathring{\pi}_n$. This choice is designed to give definite meaning to $\mathring{\pi}_n(k)$ even at k for which eigenvalues are absorbed into or emerge from the essential spectrum. See Figure 1 for an illustration of the complications.

Proposition 2.12. *We have that $\mathring{\mathcal{C}} \subset D(Y) \cap \int_{\mathbb{T}^*} \tilde{H}^2\, \frac{dk}{|\mathbb{T}^*|}$.*

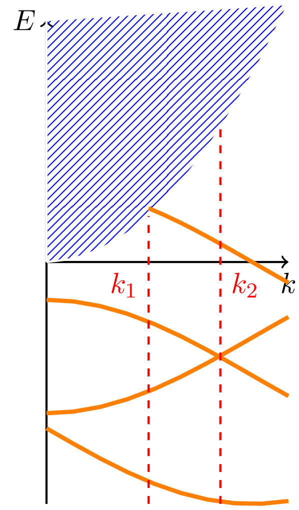

FIGURE 1. An illustration of issues with numbering the eigenvalues. In blue, we see the essential spectrum. In orange, we have the different bands. One can see that $\mathring{\pi}_4$ has a discontinuity at k_1 and $\mathring{\pi}_2$ has a jump in rank at k_2.

Proof. The fact that $\mathring{\mathcal{C}} \subset D(Y)$ follows from the smoothness in the definition and Proposition 2.6. That $\mathring{\mathcal{C}} \subset \int_{\mathbb{T}^*}^{\oplus} \tilde{H}^2 \frac{dk}{|\mathbb{T}^*|}$ is a consequence of the fact that for any k the range of each $\mathring{\pi}_n(k)$ consists of eigenfunctions of $H(k)$, which therefore must be in its domain $\tilde{H}^2$. □

Next, we will prove that, away from the essential spectrum, the eigenvalues of $H(k)$ are not constant as functions of k. This lemma is based on the classic proof of Thomas [38] (and its simplification in [37]), showing that periodic Schrödinger operators have purely absolutely continuous spectrum. See also [33].

Lemma 2.13. *Let $U \subset \mathbb{T}^*$ be an open neighborhood and $E : U \to \mathbb{R}$ be an analytic function so that for all $k \in U$, $E(k)$ is an eigenvalue of $H(k)$ below $\|k\|^2$. Then, E is not a constant function.*

Remark 2.14. This may be inferred from the main result of [16], but we supply a more direct proof.

Proof. Assume to the contrary that $E(k) = E$ for some $E \in \mathbb{R}$ and all $k \in U$. Choose some $k^* \in U$ and I, an open interval containing 0, so that $k^* + I\mathbf{e}_1 \subset U$. We consider $\tilde{H}_0(z)$ the extension of $H_0(k)$ to the strip $\mathbb{V} = \{z \in \mathbb{C} \mid \Re z \in I\}$ that is defined via

$$\tilde{H}_0(z) = -\Delta_x + (i\nabla_y - k^* + z\mathbf{e}_1)^2.$$

Using the shorthands

$$\frac{j}{L} = \sum_{u=1}^{m} \frac{j_u}{L_u}\mathbf{e}_u, \ \frac{j}{L}y = \sum_{u=1}^{m} \frac{j_u}{L_u} y_u,$$

we have that for $u(x, y) \in \tilde{H}^2$

$$\mathcal{F}_y\mathcal{F}_x\tilde{H}_0(z)\mathcal{F}_x^{-1}\mathcal{F}_y^{-1}u = \sum_{j\in\mathbb{Z}^m} \left(\|\xi\|^2 + \|2\pi\frac{j}{L} - k + iz\mathbf{e}_1\|^2\right) \hat{u}_n(\xi)e^{i2\pi \frac{j}{L}y},$$

where $\mathcal{F}_x$ is the Fourier transform in the x variable, $\mathcal{F}_y$ is the Fourier transform in y, and $\hat{u}_n(\cdot)$ is the nth Fourier coefficient of $\mathcal{F}_x u$. Now, for any $t \in \mathbb{R}$, the imaginary part of $\|\xi\|^2 + \|\frac{2\pi}{L}n - k^* + i\mathbf{e}_j t\|^2$

is uniformly bounded below by $c|t|$, yielding the estimate

$$\|\tilde{H}_0(it)u\| \geq C|t|\|u\|$$

for some constant $C > 0$. It follows then that

$$\|(\tilde{H}_0(it) - E)u\| \geq (C|t| - |E|)||u||, \tag{2.3.1}$$

and therefore, for t large enough

$$\|V(\tilde{H}_0(t) - E)^{-1}\|_{\mathrm{op}} \leq \frac{\|V\|_\infty}{C|t| - |E|},$$

so that for all t sufficiently large, -1 is not an eigenvalue of $V(\tilde{H}_0(it) - E)^{-1}$. We will use this to show that -1 is only an eigenvalue of $V(\tilde{H}_0(z) - E)^{-1}$ for finitely many points in $\mathbb{V}$. We let D denote the set

$$D = \{z \in \mathbb{V} \mid E \notin \sigma(\tilde{H}_0(z))\}.$$

For $z \in D$ we have that $z \mapsto V(H_0(z) - E)^{-1}$ is a holomorphic family of compact operators (as V is relatively compact to $\tilde{H}_0(z)$ by Proposition 2.3) and for such a family, by [26, Thm 1.9, p.370], any number is either an eigenvalue for all $z \in D$ or at finitely many points inside any compact set. We have shown that -1 cannot be an eigenvalue of $\tilde{H}_0(z)$ for every z so the latter case must hold. Note that, because $E < \|k\|^2$ for all $k \in U$, we have that $k^* + \mathbf{e}_1 I \subset D$ and therefore the set of points $s \in I$ such that -1 is an eigenvalue of $V(\tilde{H}_0(s) - E)^{-1} = V(H_0(k + s\mathbf{e}_1) - E)^{-1}$ is finite. It follows that $\mathrm{Id} + V(H_0(k) - E)^{-1}$ and $H_0(k) - E$ are invertible for all but finitely many points in $k^* + \mathbf{e}_1 I$ and therefore

$$H(k) - E = (\mathrm{Id} + V(H_0(k) - E)^{-1})(H_0(k) - E)$$

is also invertible away from finitely many points. This contradicts the assumption that E is an eigenvalue of $H(k)$ for all $k \in U$ so we are done. □

We are now ready to prove one part of Theorem 1.6, namely that states in $\mathring{\mathcal{C}}$ undergo ballistic transport in the y-directions. Recall that $Y_H(T) = e^{iTH} Y e^{-iTH}$ is the Heisenberg-evolved position operator in the y-directions.

Theorem 2.15. *For any $\psi \in \mathring{\mathcal{C}} \setminus \{0\}$, the* asymptotic velocity

$$\lim_{T \to \infty} \frac{1}{T} Y_H(T)\psi \tag{2.3.2}$$

exists and is non-zero.

Proof. Let $P^y = -i\nabla_y$ be the momentum operator in the y-direction. Making use of the identity

$$Y_H(T)\psi = Y_H(0)\psi + 2\int_0^T P_H^y(t)\psi \, dt,$$

valid for $\psi \in H^1(\mathbb{R}^{n+m}) \cap D(Y)$, it is enough to show that $\lim_{T\to\infty} \frac{1}{T}\int_0^T P_H^y(t)\psi \, dt$ exists and is non-zero.

Writing $U\psi$ for $\psi \in \mathring{\mathcal{C}}$ as

$$U\psi = \int_{\mathbb{T}^*}^{\oplus} \sum_{n=1}^{M} \mathring{\pi}_n(k) U\psi \, \frac{dk}{|\mathbb{T}^*|} \tag{2.3.3}$$

and applying the partial Floquet transform to P^k_H, we find

$$U\frac{1}{T}\int_0^T P^y_H(t)\psi\,dt = \frac{1}{T}\int_0^T\int_{\mathbb{T}^*}^{\oplus} e^{itH(k)}\left(P^y+k\right)e^{-itH(k)}U\psi\;\frac{dk}{|\mathbb{T}^*|}dt. \tag{2.3.4}$$

Now let $\mathcal{P}_{\mathrm{pp}}(k)$ and $\mathcal{P}_{\mathrm{c}}(k)$ be the projectors onto the pure point and continuous subspaces of $H(k)$, respectively, and write (2.3.4) as

$$\begin{aligned}
&\frac{1}{T}\int_0^T\int_{\mathbb{T}^*}^{\oplus} e^{itH(k)}(P^y+k)\sum_{n=1}^{M} e^{-itE_n(k)}\mathring{\pi}_n(k)U\psi(k,\cdot,\cdot)\;\frac{dk}{|\mathbb{T}^*|}\,dt\\
&=\frac{1}{T}\int_0^T\int_{\mathbb{T}^*}^{\oplus}\sum_{n=1}^{N} e^{it(H(k)-E_n(k))}(\mathcal{P}_{\mathrm{pp}}(k)+\mathcal{P}_{\mathrm{c}}(k))\left(P^y+k\right)\mathring{\pi}_n(k)U\psi(k,\cdot,\cdot)\;\frac{dk}{|\mathbb{T}^*|}dt\\
&=\frac{1}{T}\int_0^T\int_{\mathbb{T}^*}^{\oplus}\sum_{m=1}^{N(k)}\sum_{n=1}^{M} e^{it(E_m(k)-E_n(k))}\mathring{\pi}_m(k)\left(P^y+k\right)\mathring{\pi}_n(k)U\psi(k,\cdot,\cdot)\;\frac{dk}{|\mathbb{T}^*|}dt\\
&+\frac{1}{T}\int_0^T\int_{\mathbb{T}^*}^{\oplus}\sum_{n=1}^{M} e^{it(H(k)-E_n(k))}\mathcal{P}_{\mathrm{c}}(k)\left(P^y+k\right)\mathring{\pi}_n(k)U\psi(k,\cdot,\cdot)\;\frac{dk}{|\mathbb{T}^*|}dt\\
&:= A(T)+B(T)
\end{aligned}$$

We will show below that

$$\lim_{T\to\infty}A(T)=\int_{\mathbb{T}^*}^{\oplus}\sum_{n=1}^{M}\mathring{\pi}_n(k)\left(P^y+k\right)\mathring{\pi}_n(k)U\psi(k,\cdot,\cdot)\,\frac{dk}{|\mathbb{T}^*|},$$

that $B(T)\to 0$, and finally that the expression for the limit of $A(T)$ is non-zero. The proof of this first fact is an argument from [1], but for completeness, we add the details below:

$$\begin{aligned}
&\lim_{T\to\infty}\Bigg\|\frac{1}{T}\int_0^T\int_{\mathbb{T}^*}^{\oplus}\sum_{m=1}^{N(k)}\sum_{n=1}^{M} e^{it(E_m(k)-E_n(k))}\mathring{\pi}_m(k)\left(P^y+k\right)\mathring{\pi}_n(k)U\psi(k,\cdot,\cdot)\,\frac{dk}{|\mathbb{T}^*|}dt\\
&\quad-\int_{\mathbb{T}^*}^{\oplus}\sum_{n=1}^{M}\mathring{\pi}_n(k)\left(P^y+k\right)\mathring{\pi}_n(k)U\psi(k,\cdot,\cdot)\,\frac{dk}{|\mathbb{T}^*|}\Bigg\|^2\\
&\quad=\int_{\mathbb{T}^*}^{\oplus}\left\|\sum_{m=1}^{N(k)}\sum_{\substack{n\neq m\\ n=1}}^{M}\frac{1}{T}\int_0^T e^{it(E_m(k)-E_n(k))}dt\,\mathring{\pi}_m(k)\left(P^y+k\right)\mathring{\pi}_n(k)U\psi(k,\cdot,\cdot)\right\|^2\;\frac{dk}{|\mathbb{T}^*|}\\
&\quad=\int_{\mathbb{T}^*}^{\oplus}\sum_{m=1}^{N(k)}\left\|\mathring{\pi}_m(k)\sum_{\substack{n\neq m\\ n=1}}^{M}\frac{1}{T}\int_0^T e^{it(E_m(k)-E_n(k))}dt\left(P^y+k\right)\mathring{\pi}_n(k)U\psi(k,\cdot,\cdot)\right\|^2\;\frac{dk}{|\mathbb{T}^*|}
\end{aligned}$$

by the Pythagorean theorem. We observe that by the definition of the $E_n(k)$, the integrand is $O(1/T)$ for each k. So, by the triangle inequality and squaring

$$\begin{aligned}&\left\|\mathring{\pi}_m(k)\sum_{n=1}^{M}\frac{1}{T}\int_0^T e^{it(E_m(k)-E_n(k))}dt\ (P^y+k)\,\mathring{\pi}_n(k)U\psi(k,\cdot,\cdot)\right\|^2\\&\leq M^2\max_{1\leq n\leq M}\|\mathring{\pi}_m(k)\,(P^y+k)\,\mathring{\pi}_n(k)U\psi(k,\cdot,\cdot)\|^2,\end{aligned}$$

we may conclude via the dominated convergence theorem.

We now show the limit of $B(T)$ is 0. We have

$$\|B(T)\|^2=\int_{\mathbb{T}^*}^{\oplus}\left\|\sum_{n=1}^{M}\frac{1}{T}\int_0^T e^{it(H_{\mathrm{c}}(k)-E_n(k))}\,dt\ (P^y+k)\,\mathring{\pi}_n(k)U\psi(k,\cdot,\cdot)\right\|^2\frac{dk}{|\mathbb{T}^*|},$$

and denoting $\tilde{\psi}_n^k(x,y)=\mathring{\pi}_n(k)U\psi(k,x,y)$ we can write the above as

$$\begin{aligned}&\sum_{n,m=1}^{M}\frac{1}{T^2}\left\langle\int_0^T e^{it(H(k)-E_n(k))}\mathrm{d}t\,\mathcal{P}_{\mathrm{c}}\,(P^y+k)\,\tilde{\psi}_n^k,\int_0^T e^{is(H(k)-E_m(k))}\mathcal{P}_{\mathrm{c}}ds\ (P^y+k)\,\tilde{\psi}_m^k\right\rangle\\&=\sum_{n,m=1}^{M}\frac{1}{T^2}\iint_{[0,T]\times[0,T]}e^{i(tE_n-sE_m)}\,\langle\mathcal{P}_{\mathrm{c}}\,(P^y+k)\,\tilde{\psi}_n^k,e^{iH(k)(s-t)}\mathcal{P}_{\mathrm{c}}\,(P^y+k)\,\tilde{\psi}_m^k\rangle\,ds\,dt\end{aligned}$$

Now use the spectral theorem to see that

$$\begin{aligned}&\frac{1}{T^2}\iint_{[0,T]\times[0,T]}e^{i(sE_n-tE_m)}\,\langle\mathcal{P}_{\mathrm{c}}\,(P^y+k)\,\tilde{\psi}_n^k,e^{iH(k)(s-t)}\mathcal{P}_{\mathrm{c}}\,(P^y+k)\,\tilde{\psi}_m^k\rangle\,ds\,dt\\&=\frac{1}{T^2}\iint_{[0,T]\times[0,T]}e^{i(sE_n-tE_m)}\,ds\,dt\int_{\mathbb{R}}e^{i(s-t)\lambda}\mu(d\lambda)\,ds\,dt,\end{aligned}$$

where μ is the spectral measure of $\mathcal{P}_{\mathrm{c}}\,(P^y+k)\,\tilde{\psi}_n^k$ and $\mathcal{P}_{\mathrm{c}}\,(P^y+k)\,\tilde{\psi}_m^k$, which is, due to the projections, a continuous measure. As in the proof of Wiener's theorem, we use Fubini's theorem to rewrite the integral as

$$\int_{\mathbb{R}}\frac{1}{T^2}\iint_{[0,T]\times[0,T]}e^{is(E_n-\lambda)}e^{it(\lambda-E_m)}\,ds\,dt\,\mu(d\lambda)$$

and observe that the inner integral goes to $\chi_{\{0\}}(E_n-\lambda)\chi_{\{0\}}(E_m-\lambda)$. Since the integrand is uniformly bounded, we may use the dominated convergence theorem to find

$$\begin{aligned}\lim_{T\to\infty}\|B(T)\|^2&=\lim_{T\to\infty}\int_{\mathbb{R}}\frac{1}{T^2}\sum_{n,m=1}^{M}\iint_{[0,T]\times[0,T]}e^{is(E_n-\lambda)}e^{it(\lambda-E_m)}\,ds\,dt\,\mu(d\lambda)\\&=\mu(\{E_n\}\cap\{E_m\})=0,\end{aligned}$$

as claimed.

This justifies that the desired limit exists, whereas to see that it is non-zero, we use the identity

$$\mathring{\pi}_n(k)\,(P^y+k)\,\mathring{\pi}_n(k)U\psi(k,\cdot,\cdot)=\left(\frac{1}{2}\nabla E_n\right)\mathring{\pi}_n(k)U\psi(k,\cdot,\cdot)$$

valid for $k \in \operatorname{supp} \mathring{\pi}_n(k)U\psi$. This identity can be derived by using that, on the support of $\mathring{\pi}_n U\psi$ we may differentiate $\mathring{\pi}_n(k)H(k)\mathring{\pi}_n(k)$. Doing this in two ways, we find that

$$\begin{aligned}
\nabla_k(\mathring{\pi}_n(k)H(k)\mathring{\pi}_n(k)) &= \nabla_k\mathring{\pi}_n(k)(H(k)\mathring{\pi}_n(k)) + \mathring{\pi}_n(k)(2(P^y+k))\mathring{\pi}_n(k) + \mathring{\pi}_n(k)H(k)\nabla_k\mathring{\pi}_n(k) \\
&= \mathring{\pi}_n(k)(2(P^y+k))\mathring{\pi}_n(k) + E_n(k)\left(\nabla_k\mathring{\pi}_n(k)\mathring{\pi}_n(k) + \mathring{\pi}_n(k)\nabla_k\mathring{\pi}_n(k)\right) \\
&= \pi_n(k)(2(P^y+k))\mathring{\pi}_n(k) + E_n(k)\nabla_k\mathring{\pi}_n(k)
\end{aligned}$$

and since $\mathring{\pi}_n^2(k) = \mathring{\pi}_n(k)$, we also have

$$\nabla(\mathring{\pi}_n(k)H(k)\mathring{\pi}_n(k)) = \nabla_k(E_n(k)\mathring{\pi}_n(k)) = \nabla_k E_n(k)\mathring{\pi}_n(k) + E_n(k)\nabla_k\mathring{\pi}_n(k).$$

Thus, we may compute

$$\begin{aligned}
&\left\| \int_{\mathbb{T}^*}^{\oplus} \sum_{n=1}^{M} \mathring{\pi}_n(k)\,(P^y+k)\,\mathring{\pi}_n(k)U\psi(k,\cdot,\cdot)\,\frac{dk}{|\mathbb{T}^*|} \right\|^2 \\
&= \int_{\mathbb{T}^*}^{\oplus} \sum_{n=1}^{M} \|\mathring{\pi}_n(k)\,(P^y+k)\,\mathring{\pi}_n(k)U\psi(k,\cdot,\cdot)\|^2\,\frac{dk}{|\mathbb{T}^*|} \\
&= \frac{1}{2}\int_{\mathbb{T}^*} \sum_{n=1}^{M} |\nabla_k E_n(k)|^2\,\|\mathring{\pi}_n(k)U\psi(k,\cdot,\cdot)\|^2\,\frac{dk}{|\mathbb{T}^*|} > 0,
\end{aligned}$$

where the final inequality follows from Lemma 2.13.

□

Remark 2.16. It is natural to ask whether the above result could be extended to V partially in the Pastur-Tkachenko class of *limit*-periodic operators. The first step towards this result would be a quantitative version of the convergence part of this theorem. However, [41] relies on the Hill discriminant for this estimate, and it is unclear how to proceed in this setting where the discriminant is no longer available. See also [12] for the discrete version of this result, relying instead on a product formula for the Radon-Nikodym derivative of the density of states measure.

We will now prove the following result regarding the absence of transport in the x-directions. We note that this result holds for all surface states, not just those in $\mathring{\mathcal{H}}_{\mathrm{sur}}$.

Proposition 2.17. *Let $X_H(T) = e^{iTH}Xe^{-iTH}$ be the Heisenberg-evolved position operator in the x-directions. For any $\psi \in \mathcal{H}_{\mathrm{sur}} \cap D(X) \cap H^2(\mathbb{R}^{n+m})$ the asymptotic velocity in the x-directions vanishes, i.e.*

$$\lim_{T\to\infty} \frac{1}{T} X_H(T)\psi = 0.$$

Proof. Let $P^x = -i\nabla_x$ be the momentum operator in the x-directions. Making use of the identity

$$X_H(T)\psi = X_H(0)\psi + 2\int_0^T P_H^x(t)\psi\,dt,$$

valid for $\psi \in H^1(\mathbb{R}^{n+m}) \cap D(X)$, it is enough to show that $\frac{1}{T}\int_0^T P_H^x(t)\psi\,dt \to 0$ as $T \to \infty$. Since $\psi \in \int_{\mathbb{T}^*}^{\oplus} \mathcal{H}_{\mathrm{pp}}(k)\,\frac{dk}{|\mathbb{T}^*|}$ we can write

$$U\psi(k,x,y) = \sum_{n=1}^{\infty} a_n(k)\varphi_n(k,x,y).$$

Let $\varepsilon > 0$. Then for each k there is $N(k)$ such that if we let

$$\psi_N(k,x,y) = \sum_{n=1}^{N(k)} a_n(k)\varphi_n(k,x,y),$$

then we will have that

$$\|U\psi(k,\cdot,\cdot) - \psi_N(k,\cdot,\cdot)\|_{\tilde{H}^2} < \varepsilon$$

uniformly in k. Then we can write

$$\begin{aligned}\left\| \frac{1}{T}\int_0^T P_H^x(t)\psi\, dt - \frac{1}{T}\int_0^T P_H^x(t)U^*\psi_N dt \right\| &\le \frac{1}{T}\int_0^T \|P_H^x(t)(\psi - U^*\psi_N)\|\, dt \\ \le \frac{1}{T}\int_0^T \|P_H^x(t)(H-i)^{-1}\|_{\mathrm{op}}\|(H-i)(\psi - U^*\psi_N)\|\, dt.\end{aligned}$$

We note that $P_H^x(t)(H-i)^{-1}$ is uniformly bounded (as P_H^x is H_0 relatively bounded) and furthermore

$$\|(H-i)(\psi - U^*\psi_N)\|^2 = \int_{\mathbb{T}^*}^{\oplus} \|(H(k)-i)(U\psi - \psi_N)\|^2 \frac{dk}{|\mathbb{T}^*|} < \varepsilon^2.$$

We conclude that it is enough to prove $\frac{1}{T}\int_0^T P_H^x(t)U^*\psi_N\, dt \to 0$ as $T\to\infty$.

Now we note that

$$\left\| \frac{1}{T}\int_0^T P_H^x(t)U^*\psi_N\, dt \right\|^2 = \frac{1}{T^2}\int_0^T\int_0^T \langle P_H^x(s)U^*\psi_N, P_H^x(t)U^*\psi_N\rangle\, dtds.$$

Since P^x respects the periodicity in y, applying the partial Floquet transform yields

$$\begin{aligned}&\frac{1}{T^2}\iint_{[0,T]^2} \langle P_H^x(s)U^*\psi_N, P_H^x(t)U^*\psi_N\rangle\, dt\, ds \\ &= \frac{1}{T^2}\iint_{[0,T]^2}\int_{\mathbb{T}^*}^{\oplus} \langle e^{isH(k)}P^x e^{-isH(k)}\psi_N, e^{itH(k)}P^x e^{-itH(k)}\psi_N\rangle \frac{dk}{|\mathbb{T}^*|}\, dt\, ds \\ &= \int_{\mathbb{T}^*}^{\oplus} \frac{1}{T^2}\iint_{[0,T]^2} \langle e^{isH(k)}P^x e^{-isH(k)}\psi_N, e^{itH(k)}P^x e^{-itH(k)}\psi_N\rangle\, dt\, ds \frac{dk}{|\mathbb{T}^*|}\end{aligned}$$

by Fubini's theorem. By Lemma A.2, the k-integrand goes to 0 as $T\to\infty$ so we may conclude, using the dominated convergence theorem, that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T P_H^x(t)U^*\psi_N\, dt = 0,$$

as needed. □

2.4. **Proof of Theorem 1.6: density of $\mathring{\mathcal{C}}$ when m=1.** In this section, we specialize to one periodic direction, i.e., $m = 1$. By Theorem 2.15 and Proposition 2.17, we know that states in $\mathring{\mathcal{C}} \cap D(X)$ undergo directional ballistic transport. Thus, to prove Theorem 1.6, all that remains is to show that $\mathring{\mathcal{C}} \cap D(X)$ is dense in $\mathring{\mathcal{H}}_{\mathrm{sur}}$. Since $\mathring{\mathcal{C}}$ consists of smooth functions of k, this requires showing that the singularities of the projectors $\mathring{\pi}_n(k)$ are mild enough to be approximated by smooth functions. When $m = 1$, we show this using standard perturbation theory, though more sophisticated techniques should work for $m > 1$; see Remark 1.7.

Proposition 2.18. *Let $m = 1$. For every $k \in S_n$, there is a neighborhood $\mathcal{N} \subset S_n$ so that $\mathring{\pi}_n(k)$ is smooth on $\mathcal{N}$ except for at finitely many points.*

Proof. By the definition of $\mathring{\pi}_n$, the eigenvalue associated to $\mathring{\pi}_k$ is strictly less than $\|k\|^2$. Thus, the statement is just a result of the standard perturbation theory of isolated eigenvalues for a holomorphic self-adjoint family depending on one parameter (as $m = 1$). See, for instance, the results in [26], specifically, the discussion at the beginning of Part VII, Chapter 2. In particular, in $\mathcal{N}$, a neighborhood of k, we may parameterize the eigenvalues below $\|k\|^2$ and the corresponding eigenfunctions analytically. For any fixed n, $\mathring{\pi}_n(k)$ is then smooth so long as the eigenvalues below n do not cross, at which point $\mathring{\pi}_n(k)$ will experience a discontinuity. By the identity principle, the crossings may only accumulate at the boundary of $\mathcal{N}$, so by restricting we may take the discontinuities within $\mathcal{N}$ to be finite. □

Proof of Theorem 1.6. As explained above, we need only show that $\mathring{\mathcal{C}} \cap D(X)$ is dense in $\mathring{\mathcal{H}}_{\mathrm{sur}}$. First we show that $\mathring{\mathcal{C}}$ is itself dense in $\mathring{\mathcal{H}}_{\mathrm{sur}}$. For this, note that any $\psi \in \mathring{\mathcal{H}}_{\mathrm{sur}}$ satisfies

$$U\psi = \int_{\mathbb{T}^*}^{\oplus} \sum_{n=1}^{N(k)} \mathring{\pi}_n(k) U\psi \, \frac{dk}{|\mathbb{T}^*|}$$

for $N(k) \leq \infty$. Since, for any M,

$$\left\| U\psi - \int_{\mathbb{T}^*}^{\oplus} \sum_{n=1}^{M} \mathring{\pi}_n(k) U\psi \right\|^2 = \int_{\mathbb{T}^*}^{\oplus} \sum_{n=M}^{\infty} \|\mathring{\pi}_n(k) U\psi\|^2 \, dk \leq \|\psi\|^2,$$

which goes to 0 as $M \to \infty$, by the dominated convergence theorem and using our convention that $\mathring{\pi}_n(k) = 0$ for $n > N(k)$, we may approximate $U\psi$ by a sum of the form

$$\sum_{n=1}^{M} \mathring{\pi}_n(k) U\psi,$$

where $M < \infty$ is independent of k. By Proposition 2.18, we may find a countable locally finite cover of S_n by neighborhoods $\{\mathcal{N}_m\}_{m \in \mathbb{N}}$ so that except for finitely many points of each $\mathcal{N}_m$, $\mathring{\pi}_n(k)$ is smooth for all $n \leq M$. Forming a partition of unity subordinate to this cover and approximating $U\psi$ on each $\mathcal{N}_m$ in L^2 by a smooth function vanishing on the singularities now shows the result.

To conclude, we show that $D(X)$ is dense in $\mathring{\mathcal{C}}$. Any $\psi \in \mathring{\mathcal{C}}$ satisfies

$$U\psi = \sum_{n=1}^{M} \mathring{\pi}_n(k) U\psi.$$

Since the sum is finite, it is enough to show that for each summand $U^* \mathring{\pi}_n(k) U\psi$, we can approximate arbitrarily closely by a function $\tilde{\psi} \in D(X)$. By the above, because $E_n(k)$ is analytic wherever $\mathring{\pi}_n(k)$ is, we may assume that each $\mathring{\pi}_n(k) U\psi$ is supported on a set where $E_n(k)$ is analytic. Furthermore, we only need to show that this approximation holds for any $I \subset S_n$ an open interval because, as above, this local statement suffices by a partition of unity argument.

Now, on this interval I, we may assume that the set

$$\left\{k \in I \mid E_n(k) = \left\|k + 2\pi \frac{j}{L}\right\|^2, j \in \mathbb{Z}\right\}$$

is either finite or all of I by the identity principle since we are free to restrict I further to prevent accumulation on the boundary. In the first case, the quantity

$$\delta(k) = \min_{j\in\mathbb{Z}} \left\{ \left\|k + 2\pi \frac{j}{L}\right\|^2 - E_n(k) \middle| \left\|k + 2\pi \frac{L}{j}\right\|^2 - E_n(k) > 0 \right\}$$

will be uniformly bounded below on the complement of any ε-neighborhood containing these points. In the latter case, we will have $E_n(k) = \|k + 2\pi \frac{j_0}{L}\|^2$ for some $j_0 \in \mathbb{Z}$, but then $\mathring{\pi}_n U\psi$ will be 0 by the definition of $\mathring{\mathcal{H}}_{\mathrm{sur}}$.

Thus, we denote this set of points by $\{p_i\}_{i=1}^N$. For $\varepsilon > 0$, let χ_ε be a smooth function, supported on I, which is 0 on $(p_i - \varepsilon, p_i + \varepsilon)$ and 1 on $I \setminus (p_i - 2\varepsilon, p_i + 2\varepsilon)$. Now consider $\tilde{\psi}_{n,I} = \chi_\varepsilon(k)\mathring{\pi}_n(k)U\psi$, which is smooth because $\psi \in \mathring{\mathcal{C}}$ and can be made arbitrarily close to $\mathring{\pi}_n(k)U\psi$ by taking ε small enough. For any $\varepsilon > 0$, we have that $\delta(k)$ is uniformly bounded below away from 0 on the support of $\tilde{\psi}_{n,I}$. Then, by Proposition 2.8 we have that

$$\int_{\mathbb{T}^*} \|X\tilde{\psi}_{n,I}(k)\|^2 \frac{dk}{|\mathbb{T}^*|} \le C_\delta^2 \int_I \|\tilde{\psi}_{n,I}(k)\|^2 \frac{dk}{|\mathbb{T}^*|} < \infty.$$

Therefore, $\tilde{\psi}_{n,I}$ is in $D(X)$ so the proof is complete. □

3. Ballistic transport for the scattering states

In this section, we will show that we have a dense set of states $\mathcal{D}$ such that for $\psi \in \mathcal{D}$, $\Omega\psi$ exhibits ballistic transport. This proof will be presented simultaneously on $\mathbb{R}^{n+m}$ and $\mathbb{Z}^{n+m}$ with no restriction on m. In fact, the periodicity plays no role in this section; only the compact support in the x-directions matters, as one expects from such scattering arguments.

On $\mathbb{R}^{n+m}$ define

$$\mathcal{D}_a = \mathrm{Span}(\{\psi_x \otimes \psi_y \in \mathcal{S}(\mathbb{R}^n) \otimes \mathcal{S}(\mathbb{R}^m) \mid \mathrm{supp}\, \hat{\psi}_x \Subset B_a^c\}),$$

where the span is finite, B_a^c is the complement of the ball of radius $a > 0$ and center 0 in $\mathbb{R}^n$, and $\mathcal{S}$ is the space of Schwartz functions. On $\mathbb{Z}^{n+m}$ define

$$\mathcal{D}_a = \mathrm{Span}(\{\psi_x \otimes \psi_y \in \ell^2(\mathbb{Z}^n) \otimes \ell^2(\mathbb{Z}^m) \mid \hat{\psi}_x \otimes \hat{\psi}_y \in C^\infty(\mathbb{T}^{n+m}) \text{ and } \mathrm{supp}\, \hat{\psi}_x \Subset \tilde{B}_a^c\}),$$

where

$$\tilde{B}_a^c = \{k \in \mathbb{T}^* \mid |\sin(k_j)| > a, \forall 1 \le j \le n\}.$$

In both cases we define $\mathcal{D} = \bigcup_{a>0} \mathcal{D}_a$, which is dense because both B_a^c and $\tilde{B}_a^c$ exhaust $\mathbb{R}^n$ and $\mathbb{T}^*$, respectively.

We begin with the following propagation estimates:

Proposition 3.1. *Suppose that $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\mathrm{supp}\, \hat{\psi} \subset B_a^c$ for some $a > 0$. Then for any $\ell > 0$ there exists $C > 0$ depending only on ψ, ℓ, and a so that for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ such that $\frac{\|x\|}{|t|} < a$ we have that*

$$|e^{-itH_0}\psi(x)| < C(1 + \|x\| + |t|)^{-\ell}.$$

Furthermore, with χ_R^c the indicator function of $\{\|x\| \le R\}$ for $R > 0$, we have that

$$\|\chi_R^c e^{-itH_0}\psi\| \le C(1 + |t|)^{-\ell}$$

uniformly for t and R satisfying $\frac{R}{|t|} < a$.

Proof. The first inequality is a direct consequence of the representation

$$e^{-itH_0}\psi(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i(x\cdot\xi - t\xi^2)}\hat{\psi}(\xi)\, d\xi$$

and the principle of non-stationary phase, namely the Corollary to Theorem XI.14 of [34]. The second inequality is now easily seen by integrating the first. □

Proposition 3.2. *Suppose that $\psi \in \ell^2(\mathbb{Z}^n)$ is such that $\hat{\psi} \in C^\infty(\mathbb{T}^*)$ and satisfies $\operatorname{supp}\hat{\psi} \subset \tilde{B}_a^c$ for some $a > 0$. Then for any $\ell > 0$, there exists a constant $C > 0$ depending only on ψ, ℓ, and a so that for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}$ such that $\frac{\|n\|}{|t|} < a$ we have that*

$$|(e^{-it\Delta}\psi)(n)| < C(1 + \|n\| + |t|)^{-\ell},$$

and also

$$\|\chi_R^c e^{-itH_0}\psi\| \le C(1 + |t|)^{-\ell},$$

uniformly for t and R satisfying $\frac{R}{|t|} < a$.

Proof. Using the identity

$$e^{-i\omega(k)} = \frac{1}{-i\partial_1\omega}\partial_1[e^{i\omega(k)}],$$

valid when $\partial_1\omega \neq 0$, we integrate by parts ℓ times in the inversion formula

$$(e^{-itH_0}\psi)(n) = \int_{\mathbb{T}^*} e^{i(k\cdot n - 2t\sum_{j=1}^n \cos(k_j))}\hat{\psi}(k)\, \frac{dk}{|\mathbb{T}^*|}$$

to find that

$$(e^{-itH_0}\psi)(n) = \int_{\mathbb{T}^*} e^{i(k\cdot n - 2t\sum_{j=1}^n \cos(k_j))} L^\ell(\hat{\psi})\, \frac{dk}{|\mathbb{T}^*|},$$

where L is the differential operator $f(k) \mapsto \partial_1[\frac{i}{n_1 + 2t\sin(k_1)} f(k)]$. By the product rule, it follows that

$$\left|(e^{itH_0}\psi)(n)\right| \le C \max_{k\in\operatorname{supp}\hat{\psi}} |n_1 + 2t\sin(k_1)|^{-\ell}$$

for C depending on the first ℓ derivatives of $\hat{\psi}$. Now, on the support of $\hat{\psi}$ we have that

$$|n_1 + 2t\sin(k_1)| > 2|t||\sin(k_1)| - |n_1| > 2|t|a - |n_1|$$

and it is easy to check that there is some $c > 0$ such that

$$2|t|a - |n_1| > c(|n_1| + |t|)$$

uniformly for n_1 and t satisfying $\frac{|n_1|}{|t|} < a$. The first inequality now follows immediately, whereas the second now follows from summing the first inequality. □

We can now prove

Proposition 3.3. *Suppose that V is strip periodic, $V \in C^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, and $\nabla V \in L^\infty(\mathbb{R}^d)$. Then for any $\psi \in \mathcal{D}$ we have that $\Omega\psi$ exists and exhibits directional ballistic transport.*

Similarly, suppose that V is strip periodic on $\ell^2(\mathbb{Z}^d)$. Then for any $\psi \in \mathcal{D}$, we have that $\Omega\psi$ exists and exhibits ballistic transport.

Proof. We only prove the first statement because the proof of the second is similar by using the analogous bounds from Proposition 3.2 and Lemma B.2. By Proposition B.3 it is enough to show that for any $\psi \in \mathcal{D}_a$, $\ell > 0$, and t large enough, there is $C > 0$ so that

$$\|Ve^{-itH_0}\psi\|_{H_1} \le C(1+t)^{-\ell} \tag{3.0.1}$$

$$\|Ve^{-itH_0}\nabla\psi\| \le C(1+t)^{-\ell} \tag{3.0.2}$$

$$\|XVe^{-itH_0}\psi\| \le C(1+t)^{-\ell} \tag{3.0.3}$$

$$\|YVe^{-itH_0}\psi\| \le C(1+t)^{-\ell}. \tag{3.0.4}$$

Allowing $C > 0$ to change from line to line, by the definition of the H^1 norm and assuming that V is supported in χ_R^c, we have that

$$\begin{aligned}\|Ve^{-itH_0}\psi\|_{H_1} &\le C(\|Ve^{-itH_0}\psi\| + \|\nabla Ve^{-itH_0}\psi\| + \|Ve^{-itH_0}\nabla\psi\|)\\ &\le C(\|\chi_R^c e^{-itH_0}\psi\| + \|\chi_R^c e^{itH_0}\nabla\psi\|),\end{aligned}$$

where C depends on $\|V\|_{L^\infty}$ and $\|\nabla V\|_{L^\infty}$. By linearity it is enough to consider $\psi = \psi_x \otimes \psi_y$, which allows us to see that

$$\|\chi_R^c e^{-itH_0}\psi\| = \|\chi_R^c e^{-itH_0^x}\psi_x\|\|e^{-itH_0^y}\psi_y\|,$$

and similarly for $\nabla\psi$. As the components of $\nabla\psi$ are again in $\mathcal{D}_a$, from Proposition 3.1 we obtain the inequalities (3.0.1) and (3.0.2) for $t > \frac{R}{a}$.

Next, we note that

$$\|XVe^{-itH_0}\psi\| = \|X\chi_R^c Ve^{-itH}\psi\| \le R\|V\|_{L^\infty}\|\chi_R^c e^{-i\tau H_0}\psi\|,$$

so the same argument yields the inequality (3.0.3), as well. Finally, we have that

$$\|YVe^{-itH_0}\psi\| \le \|V\|_{L^\infty}\|\chi_R^c e^{-itH_0^x}\psi_x\|\|Ye^{-itH_0^y}\psi_y\|.$$

The last term in the product is bounded by $C(1+t)$ due to Lemma B.2, so again applying Proposition 3.1 yields (3.0.4), thus completing the proof. $\square$

Finally, we prove Proposition 2.4, which states that $\mathcal{H}_{\rm sur}$ is given by $\int_{\mathbb{T}^*} \mathcal{H}_{\rm pp}(k)$:

Proof of Proposition 2.4. The proof of this fact is the same in the continuum and discrete settings so we do not distinguish. For convenience, write

$$\mathcal{H}_s = \int_{\mathbb{T}^*} \mathcal{H}_{\rm pp}(k)\,dk.$$

We first show the inclusion $\mathcal{H}_s \subset \mathcal{H}_{\rm sur}$, which is similar to the proof of Proposition 6.1 in [10]. Fix some $v > 0$ and $\psi \in \mathcal{H}_s$. Since $\mathcal{H}_{\rm sur}$ is a closed subspace, it suffices by density to consider ψ such that

$$U\psi(k) = \sum_{i=1}^N c_n\phi_n(k)$$

for $\phi_n(k)$ eigenfunctions of $H(k)$ and some $N < \infty$. Because the operator χ_{vt} is periodic in y, we have that

$$\|\chi_{vt}e^{-itH}\psi\|^2 = \int_{\mathbb{T}^*}\left\|\sum_{n=1}^N c_n\chi_{vt}e^{itE_n(k)}\psi_n(k)\right\|^2 \frac{dk}{|\mathbb{T}^*|},$$

which clearly goes to 0 as $t \to \infty$.

For the opposite inclusion $\mathcal{H}_{\mathrm{sur}} \subset \mathcal{H}_s$, by Proposition 2.5 (whose proof does not depend on this one) we have already identified $\mathcal{H}_{\mathrm{s}}$ as the orthogonal complement of $\operatorname{Ran}\Omega$. Therefore, it will suffice by density to check that each $\psi \in \mathcal{H}_{\mathrm{sur}}$ is orthogonal to every $\varphi \in \Omega(\mathcal{D}_a)$ for all a.

By the definition of Ω, we may find $\tilde{\varphi} \in \mathcal{D}_a$ such that

$$\lim_{t\to\infty} \|e^{-itH}\varphi - e^{-itH_0}\tilde{\varphi}\| = 0.$$

Via Propositions 3.1 and 3.2 one has that for any $0 < v < a$

$$\lim_{t\to\infty} \|\chi_{vt}^c e^{-itH_0}\tilde{\varphi}\| = 0,$$

which implies that

$$\lim_{t\to\infty} \|\chi_{vt}^c e^{-itH}\varphi\| = 0.$$

On the other hand, by the definition of $\mathcal{H}_{\mathrm{sur}}$, for all $v > 0$

$$\lim_{t\to\infty} \|\chi_{vt} e^{-itH}\psi\| = 0.$$

Therefore, we may conclude by writing

$$\begin{aligned}
|\langle \psi, \varphi\rangle| &\le |\langle \chi_{vt} e^{-itH}\psi, e^{-itH}\varphi\rangle| + |\langle e^{-itH}\psi, \chi_{vt}^c e^{-itH}\varphi\rangle| \\
&\le \|\chi_{vt}e^{-itH}\psi\|\|\varphi\| + \|\psi\|\|\chi_{vt}^c e^{-itH}\varphi\|,
\end{aligned}$$

and then taking the limit as $t \to \infty$. □

4. Embedded surface states: discrete setting

In the previous part, we have shown that eigenvalues of $H(k)$ that are away from its essential spectrum and certain thresholds vary analytically in k. Therefore, the corresponding eigenvectors integrate to states that exhibit directional ballistic transport. In this part, we will study the eigenvalues embedded in the essential spectrum. This requires a much more delicate analysis, and we only obtain results on $\mathbb{Z}^{1+1}$.

4.1. Setting.

As above, we consider a real-valued $V \in \ell^\infty(\mathbb{Z}^{1+1})$ that is L-periodic in the y variable and supported within $[-R, R]$ in the x variable. We consider

$$H := H_0 + V,$$

where now we normalize the Laplacian $H_0 = \Delta^x + \Delta^y$ so that

$$\begin{aligned}
(\Delta^x\psi)(x, y) &= \psi(x+1, y) + \psi(x-1, y) \\
(\Delta^y\psi)(x, y) &= \psi(x, y+1) + \psi(x, y-1).
\end{aligned}$$

The partial Floquet transform is defined in the same way as before: for quasimomentum $k \in \mathbb{T}^*$ and $(x, y) \in \mathbb{Z} \times \mathbb{Z}_L$, we define

$$U\psi(k, x, y) = \sum_{m\in\mathbb{Z}} \psi(x, y + mL)e^{-ik(y+mL)}.$$

Here $\mathbb{Z}_L$ may be regarded either as the set $\{0, 1, \ldots, L-1\}$ or the integers mod L since all formulas will be invariant mod L.

For fixed $k \in \mathbb{T}^*$, we write

$$U\psi(k, x, j) = \psi_x(j) \in \mathbb{C}^L.$$

In other words, for each $x \in \mathbb{Z}$, ψ_x may be regarded as a vector in $\mathbb{C}^L$. The analogue of Proposition 2.2 follows immediately for

$$(H(k)\psi)_x = \psi_{x+1} + \psi_{x-1} + \Delta^k\psi_x + V_x\psi_x,$$

where Δ^k and V (by a slight abuse of notation) are the linear maps on $\mathbb{C}^{\mathbb{Z}_L}$ given by

$$(\Delta^k \psi)(j) = e^{ik}\psi(j+1) + e^{-ik}\psi(j-1)$$
$$(V_x\psi)(j) = V(x,j)\psi(j).$$

In particular, we have the unitary equivalence

$$UHU^* = \int_{\mathbb{T}^*}^{\oplus} H(k)\,\frac{dk}{|\mathbb{T}^*|}.$$

4.2. **Spectral theory of $H_0(k)$.** We start by recording some information about $H_0(k) = \Delta^k + \Delta^x$, which will be useful in the analysis of $H(k)$.

Proposition 4.1. *Let $v_n \in \mathbb{C}^L$ be the vector $v_n(j) = \frac{1}{\sqrt{L}}\zeta^{jn}$ for $\zeta = e^{2\pi i/L}$. Then $\{v_n\}_{n\in\mathbb{Z}_L}$ is an orthonormal eigenbasis of Δ^k with associated eigenvalues $2\cos(k+2\pi n/L)$.*

Proof. We compute

$$\begin{aligned}(H(k)v_n)(j) =&(1/\sqrt{L})(e^{i(k+2(j+1)n\pi/L)} + e^{-i(k-2(j-1)n\pi/L)})\\ =&2\cos(k+2\pi n/L)\zeta^{jn}/\sqrt{L}\end{aligned}$$

which shows that v_n is an eigenvector, as claimed. The fact that these vectors are linearly independent is not immediate because two of the expressions for the eigenvalues, $2\cos(k+2\pi n/L)$, may be equal. However, it follows from the non-singularity of the Vandermonde matrix. Indeed, we may compute

$$\det(v_1\cdots v_L) = \frac{1}{L^{L/2}}\prod_{1\le m<\ell<L}(\zeta^m - \zeta^\ell) \neq 0$$

to conclude. □

Forming the vector $\Psi_x = \psi_x \oplus \psi_{x-1}$ in $\mathbb{C}^L \oplus \mathbb{C}^L$, the equation $H(k)\psi = E\psi$ may be written via transfer matrices as

$$\Psi_{x+1} = T_0(E,k)\Psi_x,$$

where

$$T_0(E,k)\Psi_x = \Big((E-\Delta^k)\psi_x - \psi_{x-1}\Big) \oplus \psi_x$$

or as a block matrix:

$$T_0(E,k) = \begin{pmatrix} E-\Delta^k & -\mathrm{Id} \\ \mathrm{Id} & 0\end{pmatrix}.$$

Observe that if $\psi_x = a_x v_j$ for all $x \in \mathbb{Z}$, then the vector-valued difference equation (4.3.1) reduces to the scalar difference equation

$$a_{x+1} = e_j a_x - a_{x-1},$$

where $e_j = E - 2\cos(k+2\pi j/L)$.

Thus, decomposing T_0 with respect to the subspaces $V_j = \mathrm{span}\{v_j\oplus 0, 0\oplus v_j\} \subset \mathbb{C}^L\oplus\mathbb{C}^L$ we see that, up to unitary conjugation,

$$T_0(E,k) = \bigoplus_{j\in\mathbb{Z}_L}\begin{pmatrix} e_j & -1 \\ 1 & 0\end{pmatrix}. \tag{4.2.1}$$

Each summand is a 1d transfer matrix of the form

$$\begin{pmatrix} w & -1 \\ 1 & 0 \end{pmatrix},$$

whose eigenvalues are parameterized by the *Joukowsky map* $J : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, which we now recall. It is defined by $J(z) = z + \frac{1}{z}$ and admits analytic inverses $\mu^- : \mathbb{C}\backslash[-2,2] \to \mathbb{C}\backslash\overline{\mathbb{D}}$ and $\mu^+ : \mathbb{C}\backslash[-2,2] \to \mathbb{D}$ where $\mathbb{D}$ is the unit disc. By solving $J(z) = w$, we may write these functions explicitly as

$$\mu^\pm(w) = \frac{w \mp \sqrt{w^2 - 4}}{2}.$$

The root is unambiguous because $\sqrt{z^2 - 4}$ has two analytic branches on $\mathbb{C} \setminus [-2,2]$ with images lying in either $\mathbb{D}$ or $\mathbb{C} \setminus \overline{\mathbb{D}}$. When $w \in [-2,2]$, we will also write $\mu^\pm(w)$ to mean the two (not necessarily unique) solutions of $J(z) = w$, but for these w's all claims will be symmetric in $+$ and $-$ so we do not fix a convention. In this case, $\mu^\pm(w)$ both lie on $\partial\mathbb{D}$.

The eigenvector associated to $\mu^\pm(w)$ is given by $\begin{pmatrix} \mu^\pm(w) \\ 1 \end{pmatrix}$. Clearly then, for $w \notin [-2,2]$, an initial condition for the difference equation corresponding to the 1d transfer matrix is decaying at $+\infty$ if and only if it is in the eigenspace of $\mu^+(w)$ and similarly at $-\infty$. On the other hand, when $w \in [-2,2]$ there are no such solutions in either direction.

With this in mind, we associate to any $(E,k) \in \mathbb{R}^2$ the subspaces $\mathcal{V}^\pm(E,k) \subset \mathbb{C}^L \oplus \mathbb{C}^L$ of vectors which decay at $\pm\infty$ under the action of T_0. If $j \in \mathbb{Z}_L$ is such that $e_j(E,k) \notin [-2,2]$, then let $V_j^\pm(E,k) \subset \mathbb{C}^L \oplus \mathbb{C}^L$ be the eigenspaces corresponding to $\mu^\pm$. The decomposition (4.2.1) and the above analysis show that $\mathcal{V}^\pm$ is given by

$$\mathcal{V}^\pm(E,k) = \bigoplus_{\{j \in \mathbb{Z}_L | e_j(E,k) \notin [-2,2]\}} V_j^\pm(E,k).$$

Each subspace $V_j^\pm$ depends on E and k through the quantity $\mu^\pm(e_j(E,k))$, where we recall that $\mu^\pm$ is an analytic function on $\mathbb{R} \setminus [-2,2]$. Therefore, each subspace $V_j^\pm$ varies analytically in E and k (in the sense that the associated projector is an analytic operator) inside the open set $\{(E,k) \in \mathbb{R}^2 \mid e_j(E,k) \notin [-2,2]\}$.

We note that since we are projecting away from the modes for which we have $e_j \in [-2,2]$, the only place where $\mathcal{V}^\pm$ might not change analytically is when for some $j \in \mathbb{Z}_L$ we have $e_j = \pm 2$. It follows that the total subspaces $\mathcal{V}^\pm$ each vary analytically away from the curves given by $e_j(E,k) = 2$ and $e_j(E,k) = -2$, for each $j \in \mathbb{Z}_L$, across which the rank of $\mathcal{V}^\pm$ may jump. Let $\mathcal{A}$ be the union of these curves, i.e.

$$\mathcal{A} = \bigcup_{j \in \mathbb{Z}_L} \{(E,k) \in \mathbb{R}^2 \mid e_j(E,k) = 2\} \cup \{(E,k) \in \mathbb{R}^2 \mid e_j(E,k) = -2\}, \tag{4.2.2}$$

and let the open set $\mathcal{U}$ be their complement inside $\mathbb{R} \times (-\frac{\pi}{L}, \frac{\pi}{L})$. Finally, for $(E,k) \in \mathbb{R}^2$ let $\mathcal{I}^\pm(E,k) : \mathcal{V}^\pm \hookrightarrow \mathbb{C}^L \oplus \mathbb{C}^L$ be the inclusion of $\mathcal{V}^\pm$ in the full space and $\mathcal{P}^\pm(E,k) : \mathbb{C}^L \oplus \mathbb{C}^L \to \mathcal{V}^\pm \subset \mathbb{C}^L \oplus \mathbb{C}^L$ be the corresponding orthogonal projection.

In summary, we have proven the following proposition:

Proposition 4.2. *In the above notation:*

- *The eigenvalues of* $T_0(E,k)$ *are given by* $\{\mu^\pm(e_j) \mid j \in \mathbb{Z}_L\}$*.*
- *A vector* $\psi_0 \in \mathbb{C}^L \oplus \mathbb{C}^L$ *satisfies* $\{T_0(E,k)^{\pm n}\psi_0\}_{n=0}^\infty \in \ell^2(\mathbb{N})$ *if and only if* $\psi_0 \in \operatorname{Ran} \mathcal{P}^\pm(E,k)$*.*
- *The operators* $\mathcal{I}^\pm(E,k)$ *and* $\mathcal{P}^\pm(E,k)$ *are analytic in* $\mathcal{U} \subset \mathbb{R}^2$*.*

4.3. **Eigenvalue problem for** $H(k)$**.** Now we turn our attention to the eigenvalue problem for the full operator $H(k)$ for some fixed $k \in \mathbb{T}^*$. Using the transfer matrix formalism developed above, we will reduce the eigenvalue problem to a connection problem across the support of the potential, as detailed in Lemma 4.3 below.

First, write $H(k)\psi = E\psi$ as the vector-valued difference equation

$$\psi_{x+1} = (E - \Delta^k - V_x)\psi_x - \psi_{x-1}, \tag{4.3.1}$$

where $V_x = V(x, \cdot)$ thought of as an operator on $\mathbb{C}^L$. As above, using $\Psi_x = \psi_x \oplus \psi_{x-1}$ in $\mathbb{C}^L \oplus \mathbb{C}^L$, we can write (4.3.1) as

$$\Psi_{x+1} = \begin{pmatrix} E - \Delta^k - V_x & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \Psi_x.$$

Now let

$$T_V = \begin{pmatrix} E - \Delta^k - V_R & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \begin{pmatrix} E - \Delta^k - V_{R-1} & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \cdots \begin{pmatrix} E - \Delta^k - V_{-R} & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \tag{4.3.2}$$

be the transfer matrix from the left side of the support of V to the right.

With this in hand, we arrive at the key lemma:

Lemma 4.3. *Let $\tilde{\mathcal{P}}^+ = Id - \mathcal{P}^+$. Then $E \in \mathbb{C}$ is an eigenvalue of $H(k)$ if and only if the matrix*

$$A(E, k) = (\tilde{P}^+ T_V \mathcal{I}^-)(E, k)$$

has nontrivial kernel. Furthermore, the eigenvalues of $H(k)$ are given by the zeroes of

$$F(\cdot, k) = \det(A^* A(\cdot, k))$$

counted with multiplicity.

Proof. From Proposition 4.2, we see that a solution to $H(k)\psi = E\psi$ is ℓ^2 at $-\infty$ if and only if $\psi_{-R} \in \operatorname{Ran} \mathcal{I}^-$ and $\psi_R \in \operatorname{Ran} \mathcal{I}^+$. Therefore, E is an eigenvalue if and only if T_V sends a vector in $\operatorname{Ran} \mathcal{I}^-$ to a vector in $\operatorname{Ran} \mathcal{P}^+$. This is precisely the condition that $A(E, k)$ is singular, since the projector $\tilde{P}^+$ enforces that the output vector has no non-decaying component. Since F is the product of the singular values of A, the second claim follows as well. $\square$

4.4. **Analyticity of solutions.** We require the following lemma on analyticity of eigenvalues and eigenprojections away from a negligible set.

Lemma 4.4. *There exists a countable set of quasimomenta $\mathcal{S} \subset \mathbb{T}^*$ with the following property: if E_0 is an eigenvalue of $H(k_0)$ for $k_0 \in \mathbb{T}^* \setminus \mathcal{S}$, then there exists an open interval $I \ni k_0$ and an analytic function $f : I \to \mathbb{R}$ so that $f(k_0) = E_0$ and $f(k)$ is an eigenvalue of $H(k)$ for all $k \in I$. Moreover, the eigenprojector associated to $f(k)$ is analytic on I.*

Proof. First, note that for any fixed k, $E \mapsto F(E, k)$ cannot vanish identically because then $H(k)$ would have an interval of eigenvalues. Thus, we may apply Lemma C.1 to F at each zero inside $\mathcal{U} = \mathbb{R} \times \left(-\frac{\pi}{L}, \frac{\pi}{L}\right) \setminus \mathcal{A}$ (if any exist) to find a neighborhood $\mathcal{N} = I \times J$ on which the zero set of F is given by some Weierstrass polynomial g with a discriminant $D(k)$ that is not identically 0. We have, by the definition of the discriminant, that

$$\mathcal{S}_{\mathcal{N}} := \{k \in J : D(k) = 0\} = \left\{ k \in J : \exists E \in \mathbb{R} \text{ s.t. } \frac{\partial g}{\partial E}(E, k) = g(E, k) = 0 \right\}.$$

which is a countable subset of J because $D(k)$ is analytic and not identically 0. If (E_0, k_0) is a zero of F with $D(k_0) \neq 0$, then we may apply the analytic implicit function theorem to conclude.

By passing to a countable subcover of neighborhoods $\mathcal{N}$, we may set $\mathcal{S}$ to be the union over all $\mathcal{S}_{\mathcal{N}}$. This establishes the claim for zeroes of F inside $\mathcal{U}$. For zeroes of F on the curves defining $\mathcal{A}$, we simply note that these curves are piecewise analytic, so the vanishing set of F on them must consist of countably many points (which we are free to add to $\mathcal{S}$) and intervals of quasimomenta. Finally, the analyticity of the associated eigenprojectors is immediate from the fact that the eigenspaces are locally given by the kernel of the analytic matrix $A(f(k), k)$. □

Remark 4.5. As in Section 2, this statement is only local; see the discussion after Remark 2.11. Though this suffices for our purposes, one may invoke the structure theory of real analytic varieties to give a global description of the energies, as is done in [40]. However, our situation is complicated by the fact that F is not analytic up to the boundary of $\mathcal{U}$ and the amount of zeros can vary with k, whereas in the classical periodic setting they are infinite and increase to infinity. As a result, it is cumbersome to properly define the function $E_n(k)$ when it is not guaranteed to exist.

Now, we establish the non-constancy of the energies:

Lemma 4.6. *For any $E_0 \in \mathbb{R}$, the set of $k \in \mathbb{T}^*$ for which E_0 is an eigenvalue of $H(k)$ has measure 0.*

Proof. The set of $k \in \mathbb{T}^*$ such that (E_0, k) is in $\mathcal{A}$ is finite (as these are non-constant curves) so it suffices to consider k such that $(E_0, k) \in \mathcal{U}$. Fix such k_0 and let I be an interval containing k_0 so that $\{E_0\} \times I \subset \mathcal{U}$. We will show that $F(E_0, k)$ vanishes only on a set of measure 0 in I or equivalently that $A(E_0, k)$ is only singular on a set of measure 0.

Now recall that $A(E_0, k)$ depends on k through the expressions $e_j(E_0, k)$, which appears in T_V and $\mu^\pm(e_j(E_0, k))$, the latter of which appears in $\mathcal{I}^+$ and $\tilde{\mathcal{P}}^+$ for some subset of j in $\mathbb{Z}_L$. Since

$$e_j(E_0, k) = E_0 - 2\cos(k + it + 2\pi j/L),$$

and the Joukowsky maps $\mu^\pm(\cdot)$ admit analytic extensions to $\mathbb{C} \setminus [-2, 2]$, we may analytically extend $A(E_0, k)$ to the vertical strip $\mathbb{V} = \{z \mid \Re z \in I\}$ so long as the image of this strip under each e_j, $j \in \mathcal{I}$, avoids $[-2, 2]$. For $k, t \in \mathbb{R}$, $e_j(E_0, k + it)$ has imaginary part

$$\sin(k + 2\pi j/L)\sinh(t).$$

Thus, unless $k + 2\pi j/L \in \pi\mathbb{Z}$, we have that $e_j(E_0, k+it)$ has non-zero imaginary part for $t \neq 0$, and therefore avoids $[-2, 2]$. We see then that, except at finitely many points, we may extend $A(E_0, k)$ to an analytic family of operators on $\mathbb{V}$.

Now, since $A(E_0, k)$ is an analytic family on $\mathbb{V}$, to show that it is singular on a set of measure 0 in I, it suffices to show that it does not vanish identically on $\mathbb{V}$. For this, we use the structure of $A(E_0, k)$. Recalling that $A(E, k) = \tilde{\mathcal{P}}^+ T_V \mathcal{I}^-$, by expanding the product (4.3.2), we may write

$$A(E, k) = \tilde{\mathcal{P}}^+ (T_0)^{2R+1} \mathcal{I}^- + \tilde{\mathcal{P}}^+ \tilde{A} \mathcal{I}^-,$$

where $\tilde{A}$ is the sum of products, each of which contains at most $2R$ copies of T_0 in addition to terms of the form V_x, $|x| \leq R$. Since the eigenvalues of $T_0(E_0, k_0 + it)$ are given by

$$\mu^\pm(E_0 - 2\cos(k_0 + it + 2\pi j/L)),$$

we see that for large $t > 0$, the exponential growth of cosine off the real axis ensures that there exist constants $C > 0$ and $c > 0$ such that

$$||T_0(E_0, k_0 + it)||_{\text{op}} \leq Ce^{ct},$$

and consequently, for some other $C > 0$

$$\|\tilde{A}(E_0, k_0 + it)\|_{\text{op}} \leq Ce^{c2Rt},$$

by virtue of the boundedness of V.

Now observe that the image of $\mathcal{I}^-$ consists of eigenspaces of T_0 corresponding to $\mu^-(e_j)$ for $j \in \mathcal{I}$ and furthermore that these eigenspaces lie in the image of $\tilde{\mathcal{P}}^+$ because, by construction $\tilde{\mathcal{P}}^+$

corresponds to the *complement* of the eigenvalues μ^+. In other words $\tilde{\mathcal{P}}^+\mathcal{I}^- = \mathcal{I}^-$. The form of μ^+ and the growth of cosine then easily show that for large t there exists $C' > 0$ so that

$$|\mu^+(e_j(E_0, k_0 + it))| > C' e^{ct}.$$

Therefore, for any $v \in \mathbb{C}^L \oplus \mathbb{C}^L$ and t sufficiently large,

$$\|\tilde{\mathcal{P}}^+(T_0)^{2R+1}\mathcal{I}^-(E_0, k_0 + it)v\| > C' e^{c(2R+1)t}\|v\|,$$

so we have for large $t > 0$

$$\begin{aligned}\|A(E, k_0 + it)\|_{\mathrm{op}} &= \|\tilde{\mathcal{P}}^+(T_0)^{2R}\mathcal{I}^- + \tilde{\mathcal{P}}^+\tilde{A}\mathcal{I}^-\|_{\mathrm{op}} \\ &\geq \|\tilde{\mathcal{P}}^+(T_0)^{2R}\mathcal{I}^-\|_{\mathrm{op}} - \|\tilde{\mathcal{P}}^+\tilde{A}\mathcal{I}^-\|_{\mathrm{op}} > C' e^{c(2R+1)t} - C e^{c2Rt}\end{aligned}$$

and we conclude that A is non-singular for $t > 0$ sufficiently large, which completes the proof. □

With Lemma 4.4 and Lemma 4.6, we may now prove Theorem 1.3. For each $k \in \mathbb{T}^*$, we let $\pi_n(k)$ be the eigenprojector associated with the nth eigenvalue of $H(k)$ (as opposed to $\mathring{\pi}_n(k)$ we do not make any restriction on the eigenvalue). Recall that the total set of surface states is given by

$$\mathcal{H}_{\mathrm{sur}} = \int_{\mathbb{T}^*}^{\oplus} \mathcal{H}_{\mathrm{pp}}(k)\, \frac{dk}{|\mathbb{T}^*|}.$$

The dense set that exhibits transport in the y-direction is

$$\mathcal{C} := \bigcup_{M=1}^{\infty} \{\psi \in \mathcal{H}_{\mathrm{sur}} \mid U\psi = \sum_{n=1}^{M} \pi_n(k)U\psi \text{ and for all } n \leq m,\ \pi_n(k)U\psi \in C^\infty(\mathbb{T}^*)\}.$$

Proposition 4.7. *The set $\mathcal{C}$ is contained in $D(Y)$ and is dense in $\mathcal{H}_{\mathrm{sur}}$. All states in $\mathcal{C}$ exhibit ballistic transport in the y-direction.*

Proof. The proof of the first sentence is the same as for $\mathring{\mathcal{C}}$ in Section 2, the key point being that each $\pi_n(k)$ is smooth away from a discrete set. Similarly, as in the proof of Theorem 2.15, the asymptotic velocity in the y-direction exists. To see that it is non-zero, we simply note that in the representation of $\psi \in \mathcal{C}$,

$$U\psi = \sum_{n=1}^{N} \pi_n(k)U\psi.$$

If $\psi \neq 0$ we must be able to find some n so that $\pi_n(k)$ is non-zero at some $k \notin \mathcal{S}$, and therefore, by Lemma 4.4, we may find some interval $I \subset \mathbb{T}^*$ on which π_n varies analytically. Therefore, the limit is non-zero because

$$\begin{aligned}&\left\| \int_{\mathbb{T}^*}^{\oplus} \sum_{n=1}^{N'} \pi_n(k)\,(P^y + k)\,\pi_n(k)U\psi(k,\cdot,\cdot)\,\frac{dk}{|\mathbb{T}^*|} \right\|^2 \\ &= \left\| \int_I^{\oplus} \pi_n(k)\,(P^y + k)\,\pi_n(k)U\psi(k,\cdot,\cdot)\,\frac{dk}{|\mathbb{T}^*|} \right\|^2 \\ &= \int_I |\nabla E_n(k)|^2\, \|\pi_n(k)U\psi(k,\cdot,\cdot)\|^2\, \frac{dk}{|\mathbb{T}^*|} > 0,\end{aligned}$$

where the fact that $\nabla E_n(k)$ is non-zero almost everywhere in I comes from Lemma 4.6. □

Combining this result with Proposition 3.3 yields Corollary 1.5:

Proof of Corollary 1.5. By Propositions 2.4 and 2.5, we have that

$$\ell^2(\mathbb{Z}^2) = \mathcal{H}_{\mathrm{sur}} \oplus \operatorname{Ran}\Omega.$$

We have shown that a dense set of states in $\mathcal{H}_{\mathrm{sur}}$ exhibit directional ballistic transport, and therefore ballistic transport, so we need only show that a dense subset of $\operatorname{Ran}\Omega$ exhibits ballistic transport. The set $\mathcal{D}$ of Proposition 3.3 is dense and any $\psi \in \Omega\mathcal{D}$ exhibits ballistic transport. Since Ω is a partial isometry we may conclude. $\square$

Appendix A. Absence of ballistic transport for pure point states.

In this section, we generalize Simon's theorem in [36] on the absence of ballistic transport for pure point states. That theorem is for operators with only pure point spectrum whereas we extend his result to operators that may also have some continuous spectrum. While quite natural, and potentially useful in other settings, to our knowledge, this extension has not appeared in the literature.

Our result is more general than is needed for this paper, as we think it is of independent interest. In fact, in the setting above, we will just use Lemma A.2, rather than the theorem as is. First, we work on $L^2(\mathbb{R}^d)$ without any distinguished coordinate directions. Recall the position operator Q:

$$Q\psi = x\psi(x)$$

with the corresponding domain

$$D(Q) = \left\{ \psi \in L^2(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} \|x\|^2 |\psi(x)|^2 \, dx < \infty \right\}.$$

We have the following theorem:

Theorem A.1. *Let $H = -\Delta + V$ with V relatively bounded. Let $\psi \in \mathcal{H}_{\mathrm{pp}} \cap D(Q) \cap H^2(\mathbb{R}^d)$. Then we have that*

$$\lim_{T\to\infty} \left\| \frac{Q}{T} e^{-iTH} \psi \right\|^2 = 0.$$

Proof. We note that

$$Q(T)\psi = Q\psi + 2\int_0^T P(t)\psi \, dt,$$

where $P = -i\nabla$ is the momentum operator and we recall that $Q(T)$ and $P(t)$ are the Heisenberg evolved position and momentum operators, respectively. Therefore, by unitarity it is enough to show that $\frac{1}{T}\int_0^T P(t)\psi \, dt \xrightarrow{T\to\infty} 0$.

Since $P(T)(H+i)^{-1}$ is uniformly bounded, by writing $P(T) = P(T)(H+i)^{-1}(H+i)$, the fact that $\psi \in H^2$ means that it suffices by density to consider ψ of the form

$$\psi(x) = \sum_{n=1}^{N} a_n \varphi_n(x)$$

where each φ_n is an eigenfunction of eigenvalue E_n. Now, we write

$$\begin{aligned}\left\| \frac{1}{T} \int_0^T P(t)\psi \, dt \right\|^2 &= \frac{1}{T^2} \left\langle \int_0^T P(s)\psi \, ds, \int_0^T P(t)\psi \, dt \right\rangle \\ &= \sum_{n=1}^N \sum_{m=1}^N \frac{\overline{a_n} a_m}{T^2} \iint_{[0,T]\times[0,T]} e^{i(sE_n - tE_m)} \left\langle P\varphi_n, e^{i(t-s)H} P\varphi_m \right\rangle dt \, ds,\end{aligned}$$

so we may further reduce the proof to showing the following lemma:

Lemma A.2. *Let H be as above and let φ_n and φ_m be eigenfunctions of H. Then we have that*

$$\lim_{T\to\infty} \frac{1}{T^2} \iint_{[0,T]\times[0,T]} e^{i(sE_n - tE_m)} \left\langle P\varphi_n, e^{i(t-s)H} P\varphi_m \right\rangle dt \, ds = 0.$$

Furthermore, for $H(k)$ as in Section 2, we have that

$$\lim_{T\to\infty} \frac{1}{T^2} \iint_{[0,T]\times[0,T]} e^{i(sE_n(k) - tE_m(k))} \left\langle P^y\varphi_n, e^{i(t-s)H(k)} P^y\varphi_m \right\rangle dt \, ds = 0.$$

Proof. Let H be as above, then we expand $P\varphi_n$ as

$$P\varphi_n = \sum_{\ell=1}^\infty p_{n,\ell}\varphi_\ell + \mathcal{P}_c P\varphi_n$$

for $p_{n,\ell} = \langle \varphi_n, P\varphi_\ell \rangle$, and $\mathcal{P}_c$ the projection to the continuous subspace, in order to obtain

$$\begin{aligned}&\frac{1}{T^2} \iint_{[0,T]\times[0,T]} e^{i(sE_n - tE_m)} \left\langle P\varphi_n, e^{i(t-s)H} P\varphi_m \right\rangle dt \, ds = \\ &\frac{1}{T^2} \sum_{\ell=1}^\infty p_{n,\ell} p_{m,\ell} \iint_{[0,T]\times[0,T]} e^{-it(E_\ell - E_n)} e^{-is(E_m - E_\ell)} \, ds \, dt \\ &+ \frac{1}{T^2} \iint_{[0,T]\times[0,T]} e^{i(sE_n - tE_m)} \left\langle \mathcal{P}_c P\varphi_n, e^{i(t-s)H} \mathcal{P}_c P\varphi_m \right\rangle\end{aligned} \tag{A.0.1}$$

because the cross-terms vanish by orthogonality. Now we consider each summand separately.

For the first summand, if either $E_n = E_\ell$ or $E_m = E_\ell$, we may use Lemma 2.3 of [36] (see also the proof of Theorem 3.1 there) to conclude that $p_{n,\ell} = 0$ or $p_{m,\ell} = 0$ respectively. If $E_n \neq E_\ell$, we get that the t integral is $O(1)$, and the integral over s is $O(T)$, which gives the desired decay. This shows that for each ℓ the summand goes to 0. Since the integral is bounded by 1 and

$$\sum_{\ell=1}^\infty |p_{n,\ell} p_{m,\ell}| \leq \sqrt{\sum_{\ell=1}^\infty |p_{\ell,m}|^2} \cdot \sqrt{\sum_{\ell=1}^\infty |p_{\ell,m}|^2} \leq \|P\varphi_n\| \|P\varphi_m\|,$$

we may use the dominated convergence theorem to pass the limit in T under the sum in (A.0.1) to see that it is $O(\frac{1}{T})$.

Turning to the second term, we use the spectral theorem to see that

$$\begin{aligned}&\frac{1}{T^2}\iint\limits_{[0,T]\times[0,T]} e^{i(sE_n-tE_m)}\,\langle \mathcal{P}_{\mathrm{c}}P\varphi_n, e^{i(t-s)H}\mathcal{P}_{\mathrm{c}}P\varphi_m\rangle\,ds\,dt\\ &=\frac{1}{T^2}\iint\limits_{[0,T]\times[0,T]} e^{i(sE_n-tE_m)}\,ds\,dt\int\limits_{\mathbb{R}} e^{i(t-s)\lambda}\mu(d\lambda)\,ds\,dt,\end{aligned}$$

where μ is the spectral measure of $\mathcal{P}_{\mathrm{c}}P\varphi_n$ and $\mathcal{P}_{\mathrm{c}}P\varphi_m$, which is, due to the projections, continuous. As in the proof of Wiener's theorem, we use Fubini's theorem to rewrite the integral as

$$\int\limits_{\mathbb{R}}\frac{1}{T^2}\iint\limits_{[0,T]\times[0,T]} e^{is(E_n-\lambda)}e^{it(\lambda-E_m)}\,ds\,dt\,\mu(d\lambda)$$

and observe that the inner integral goes to $\chi_{\{0\}}(E_n-\lambda)\chi_{\{0\}}(E_m-\lambda)$. Since the integrand is uniformly bounded, we may use the dominated convergence theorem and the fact that μ has no atoms to conclude that the entire integral goes to 0 as $T\to\infty$, which concludes the proof.

Up to interchanging symbols, the same proof applies to $H(k)$. □

With this lemma, the proof is complete. □

The analogous theorem also holds in the discrete setting for Jacobi matrices. We let h be the operator on $\ell^2(\mathbb{Z}^d)$ defined via

$$h\psi(n)=\sum_{\|n-m\|_1=1} a_{n,m}\psi(m)+b_n\psi(n)$$

for $b\in\ell^\infty(\mathbb{Z}^d,\mathbb{R})$, $a\in\ell^\infty(\mathbb{Z}^d\oplus\mathbb{Z}^d,\mathbb{R})$, and $a_{n,m}=a_{m,n}$. We note that this operator is self-adjoint by construction.

Let N be the discrete position operator

$$N\psi(n)=n\psi(n)$$

with the corresponding domain

$$D(N)=\{\psi\in\ell^2(\mathbb{Z}^d)\mid \|N\psi\|^2=\sum_{n\in\mathbb{Z}^d}\|n\|^2|\psi(n)|^2<\infty\}.$$

Theorem A.3. *For $\psi\in D(N)\cap\mathcal{H}_{\mathrm{pp}}$ we have that*

$$\lim_{T\to\infty}\frac{N}{T}e^{-iTh}\psi=0$$

Proof. Similarly to the continuous case we let $N(t)=e^{ith}Ne^{-ith}$ be the Heisenberg evolution of N and we note that

$$N(T)\psi=N(0)\psi+\int\limits_0^T i[h,N](t)\psi\,dt$$

So, we let $\tilde{P}=i[h,N]$ be the weighted momentum operator corresponding to h, or explicitly

$$\begin{aligned}-i\tilde{P}=[h,N]\psi(n)&=\sum_{\|n-m\|_1=1} a_{n,m}\psi(m)m+b_n\psi(n)n-[\sum_{\|n-m\|_1=1} a_{n,m}\psi(m)+b_n\psi(n)]n\\ &=\sum_{j=1}^d(a_{n,n+\mathbf{e}_j}\psi(n+\mathbf{e}_j)-a_{n,n-\mathbf{e}_j}\psi(n-\mathbf{e}_j))\mathbf{e}_j.\end{aligned}$$

Therefore, by unitarity it is enough to show that $\frac{1}{T}\int\limits_0^T \tilde{P}(t)\psi\, dt \xrightarrow{T\to\infty} 0$.

As in the continuum setting, it suffices by density to consider ψ of the form

$$\psi(x) = \sum_{n=1}^{N} c_n \varphi_n(x),$$

where each φ_n is an eigenfunction of eigenvalue E_n. Furthermore, the proof again reduces to the discrete analog of Lemma A.2, namely that

$$\lim_{T\to\infty} \frac{1}{T^2} \iint\limits_{[0,T]\times[0,T]} e^{i(sE_n - tE_m)} \langle \tilde{P}\varphi_n, e^{i(t-s)H}\tilde{P}\varphi_m\rangle \, dt\, ds = 0.$$

The proof of this equality is the same as before because Lemma 2.3 of [36] holds in the discrete setting. Indeed, that lemma is formulated for h the discrete Laplacian, but it is easy to convince oneself that it extends to more general Jacobi matrices. □

Appendix B. Ballistic transport via Cook's method

In this section, we give a criterion, similar to the criterion given in Cook's method [34], to show certain states exhibit ballistic transport. This lemma gives rigor to the idea that asymptotically free states should exhibit ballistic transport. Though this result seems natural, it appears to be absent from the literature.

We recall that we say that a state $\psi \in \mathcal{H} \cap D(Q_j)$ (where Q_j is the jth component of the position operator) exhibits ballistic transport in the direction of $\mathbf{e}_j$ if we have that

$$\lim_{t\to\infty} \frac{Q_j(t)}{t}\psi$$

exists and is nonzero. We also define the momentum operators, P_j which are given by $-i\partial_{x_j}$ on $\mathbb{R}^d$ and

$$(P_j\psi)(n) = -i(\psi(n+\mathbf{e}_j) - \psi(n-\mathbf{e}_j))$$

on $\mathbb{Z}^d$.

As above, the wave operator Ω (when it exists) is defined via the strong limit

$$\Omega = \operatorname*{s-lim}_{t\to\infty} \Omega(t)$$

for

$$\Omega(t) = e^{itH}e^{-itH_0}.$$

We also recall that $\psi \in \operatorname{Ran}(\Omega)$ is called asymptotically free, and its evolution is close to the free evolution. Typically, one shows the existence of the wave operator on $\Omega\psi$ via Cook's method [34, Theorem XI.4], which involves controlling the following function of t

$$Ve^{-itH_0}\psi \tag{B.0.1}$$

in L^2. Our result is an extension of this method to settings in which one can control (B.0.1) in a stronger norm.

Following Radin-Simon [32], we define the following subspaces

$$S_j(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) \mid Q_j f \in L^2, P_j f \in L^2\}$$

and

$$S_j(\mathbb{Z}^d) = \{f \in \ell^2(\mathbb{Z}^d) \mid Q_j f \in \ell^2(\mathbb{Z}^d)\}$$

equipped with the following norms

$$\|f\|_{S_j(\mathbb{R}^d)} = \sqrt{\|f\|_{H^1}^2 + \|Q_j f\|_2^2}$$

and

$$\|f\|_{S_j(\mathbb{Z}^d)} = \sqrt{\|f\|_2^2 + \|Q_j f\|_2^2}.$$

We emphasize that whenever there is no subscript to the norm, it is simply the L^2 (or ℓ^2) norm. Before proving these facts, we record a simple technical lemma that will be used in the proofs that follow.

Lemma B.1. *Let $\psi : \mathbb{R} \to L^2(\mathbb{R}^d)$ be continuous. Then, for $s < t$,*

$$\left\| Q_j \int_s^t \psi(\tau)\, d\tau \right\| \leq \int_s^t \|Q_j \psi(\tau)\|\, d\tau, \tag{B.0.2}$$

where either side of the above inequality may be infinite.

Proof. We define

$$F_N(q_j) = |q_j| \chi_{W_N}(q_j)$$

for $W_N = \{x \in \mathbb{R}^d : |x_j| \leq N\}$ and denote by $F_N(Q_j)$ the corresponding multiplication operator. By the monotone convergence theorem, we have

$$\|Q_j \varphi\| = \lim_{N\to\infty} \|F_N(Q_j)\varphi\|$$

for any $\varphi \in L^2(\mathbb{R}^d)$. Thus,

$$\begin{aligned}\left\| Q_j \int_s^t \psi(\tau)\, d\tau \right\| &= \lim_{N\to\infty} \left\| F_N(Q_j) \int_s^t \psi(\tau)\, d\tau \right\| = \lim_{N\to\infty} \left\| \int_s^t F_N(Q_j)\psi(\tau)\, d\tau \right\| \\ &\leq \lim_{N\to\infty} \int_s^t \|F_N(Q_j)\psi(\tau)\|\, d\tau\end{aligned}$$

using the boundedness of $F_N(Q_j)$. Using the monotone convergence theorem again, (B.0.2) follows. □

We will also need this ballistic upper bound:

Lemma B.2. *Suppose that $V \in L^\infty(\mathbb{R}^d)$ and $\psi \in D(Q_j)$ for some index j. Then $e^{-itH}\psi \in D(Q_j)$ for all $t \in \mathbb{R}$ and there is some $C > 0$ so that*

$$\|Q_j e^{itH}\|_{S_j(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} < C(1 + |t|).$$

Similarly, suppose that $V \in \ell^\infty(\mathbb{Z}^d)$ and $\psi \in D(Q_j)$ for some index j. Then $e^{-itH}\psi \in D(Q_j)$ for all $t \in \mathbb{R}$ and there is some $C > 0$ so that

$$\|Q_j e^{itH}\|_{S_j(\mathbb{Z}^d)\to \ell^2(\mathbb{Z}^d)} < C(1 + |t|).$$

Proof. The first claim follows from Theorem 2.1 of [32]. The second follows immediately from the triangle inequality:

$$\|(Q_j)_H(t)\psi\| \leq \|Q_j\psi\| + \left| \int_0^t \|(P_j)_H(s)\| \|\psi\| ds \right| \leq \|Q_j\psi\| + 2|t| \|\psi\|$$

yielding the bound with $C = 2\sqrt{2}$.

□

With these results in hand, we can prove the following:

Proposition B.3. *Let $V \in C^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $\nabla V \in L^\infty(\mathbb{R}^d)$. Suppose that for some index j, $\psi \in \mathcal{D}(Q_j) \cap H^1(\mathbb{R}^d)$ satisfy*

$$\int_0^\infty (1+t)\|Ve^{-itH_0}\psi\|_{S_j(\mathbb{R}^d)}\,dt < \infty, \tag{B.0.3}$$

and also

$$\int_0^\infty (1+t)\|Ve^{-itH_0}P_j\psi\|_{L_2(\mathbb{R}^d)}\,dt < \infty. \tag{B.0.4}$$

Then we have that $\Omega\psi$ and $\Omega P_j\psi$ exist and furthermore

$$\lim_{t\to\infty} \frac{(Q_j)_H(t)}{t}\Omega\psi = 2\Omega P_j\psi.$$

In particular, we have that $\Omega\psi$ exhibits ballistic transport in the Q_j-direction.

Similarly, for $V \in \ell^\infty(\mathbb{Z}^d)$, let $\psi \in \mathcal{D}(Q_j)$ satisfy

$$\int_0^\infty (1+t)\|Ve^{-itH_0}\psi\|_{S_j(\mathbb{Z}^d)}\,dt < \infty, \tag{B.0.5}$$

and also

$$\int_0^\infty (1+t)\|Ve^{-itH_0}P_j\psi\|_{\ell^2(\mathbb{Z}^d)}\,dt < \infty. \tag{B.0.6}$$

Then we have that $\Omega\psi$ exists and furthermore

$$\lim_{t\to\infty} \frac{(Q_j)_H(t)}{t}\Omega\psi = \Omega P_j\psi. \tag{B.0.7}$$

Proof. First, we note that the assumption (B.0.3) (or (B.0.5)) shows that

$$\int_0^\infty \|Ve^{-itH_0}\psi\|\,dt < \infty,$$

which implies the existence of $\Omega\psi$ by Cook's method [34], and similarly, assumptions (B.0.4) and (B.0.6) show the existence of $\Omega P_j\psi$.

To prove (B.0.7), we will start by establishing that $\Omega\psi \in D(Q_j)$, by showing that $\{Q_j\Omega(t)\psi\}$ is Cauchy. This suffices because $\Omega(t)\psi \in D(Q_j)$ for all t and Q_j is a closed operator. By Lemma B.2 $\Omega(t)\psi \in D(Q_j)$ for any $t > 0$, so we can write for $t > s > 0$

$$\|Q_j(\Omega(t)-\Omega(s))\psi\| = \|Q_j\int_s^t e^{i\tau H}(iV)e^{-i\tau H_0}\psi d\tau\| \le \int_s^t \|Q_je^{i\tau H}Ve^{-i\tau H_0}\psi\|_{L^2}d\tau$$

$$\le \int_s^t \|Q_je^{i\tau H}\|_{S_j\mapsto L^2}\|Ve^{-i\tau H_0}\psi\|_{S_j}d\tau,$$

where S_j is either $S_j(\mathbb{R}^d)$ or $S_j(\mathbb{Z}^d)$, here and in the following.

Again by Lemma B.2, we have

$$\|Q_je^{i\tau H}\|_{S_j\mapsto L^2} \le C(1+\tau),$$

so that

$$\|Q_j(\Omega(t)-\Omega(s))\psi\|_2 \leq \int_s^t (1+\tau)\|Ve^{-i\tau H_0}\psi\|_{S_j}\, d\tau.$$

This is the tail of $\int_0^\infty (1+\tau)\|Ve^{-i\tau H_0}\psi\|_{S_j} d\tau$, which converges by assumption. Thus, the sequence is Cauchy.

Now, we note that the intertwining property $e^{-itH}\Omega = \Omega e^{-itH_0}$ implies that for $t > 0$

$$\begin{aligned}
&\|Q_j e^{-itH}\Omega\psi - Q_j e^{-itH_0}\psi\| = \|Q_j\Omega e^{-itH_0}\psi - Q_j e^{-itH_0}\psi\| = \|Q_j(\Omega - \mathrm{Id})e^{-itH_0}\psi\| \\
&= \left\| Q_j \int_0^\infty e^{isH} iV e^{-isH_0} e^{-itH_0}\psi\, ds \right\| \leq \int_0^\infty \|Q_j e^{isH}\|_{S_j \mapsto L^2} \|\|Ve^{-i(s+t)H_0}\psi\|_{S_j} ds \\
&\leq \int_0^\infty (1+s)\|Ve^{-i(s+t)H_0}\psi\|_{S_j}\, ds
\end{aligned}$$

as in the above. Then we can write:

$$\begin{aligned}
\|Q_j e^{-itH}\Omega\psi - Q_j e^{-itH_0}\psi\| &\leq \int_0^\infty (1+s)\|Ve^{-i(s+t)H_0}\psi\|_{S_j}\, ds \\
&= \int_t^\infty (1+s-t)\|Ve^{-isH_0}\psi\|_{S_j}\, ds \leq \int_t^\infty (1+s)\|Ve^{-isH_0}\psi\|_{S_j}\, ds.
\end{aligned}$$

Thus, we may conclude that

$$\lim_{t\to\infty} \|Q_j e^{-itH}\Omega\psi - Q_j e^{-itH_0}\psi\| \leq \lim_{t\to\infty} \int_t^\infty (1+s)\|Ve^{-isH_0}\psi\|_{S_j} ds = 0$$

by our assumptions.

This implies that

$$\lim_{t\to\infty} \|e^{itH} Q_j e^{-itH}\Omega\psi - e^{-itH} Q_j e^{-itH_0}\psi\| = 0,$$

so in order to establish (B.0.7), it is enough to show that $\{\frac{1}{t} e^{-itH} Q_j e^{-itH_0}\psi\}$ converges.

In the continuous setting, we use the Fourier transform to see that

$$\begin{aligned}
\frac{1}{t} e^{itH} Q_j e^{-itH_0}\psi &= \frac{1}{t} e^{itH}\mathcal{F}^{-1}(2te^{-it|\xi_j|^2}\xi_j\hat{\psi} - e^{-it|\xi_j|^2} i\partial_{\xi_j}\hat{\psi}) \\
&= e^{itH}\mathcal{F}^{-1}(2e^{-it|\xi_j|^2}\xi_j\hat{\psi}) + O_{L^2(\mathbb{R}^d)}(1/t).
\end{aligned}$$

The first term is $2\Omega(t)P_j\psi$, which we established above converges.

In the discrete setting, we have

$$\begin{aligned}
\frac{1}{t} e^{itH} Q_j e^{-itH_0}\psi &= \frac{1}{t} e^{itH}\mathcal{F}^{-1}\left(-2\sin(\xi_j) t e^{-it\sum_{\ell=1}^d 2\cos(\xi_\ell)}\hat{\psi} - e^{-it\sum_{\ell=1}^d 2\cos(\xi_\ell)} i\partial_{\xi_j}\hat{\psi} \right) \\
&= e^{itH}\mathcal{F}^{-1}\left(-2\sin(\xi_j) e^{-it\sum_{\ell=1}^d 2\cos(\xi_\ell)}\hat{\psi} \right) + O_{\ell^2(\mathbb{Z}^d)}(1/t).
\end{aligned}$$

As before, the first term is $\Omega(t)P_j\psi$. Thus, we may conclude that

$$\lim_{t\to\infty}\frac{1}{t}(Q_j)_H(t)\psi = \lim_{t\to\infty}\frac{1}{t}e^{itH}Q_je^{-itH_0}\psi = \Omega P_j\psi,$$

which is clearly non-zero because $\|\Omega P_j\psi\| = \|P_j\psi\|$ is positive for $\psi \neq 0$, $\psi \in H^1(\mathbb{R}^d)$, or $\psi \in \ell^2(\mathbb{Z}^d)$, as needed. ☐

Remark B.4. As is typical in scattering theory, one may weaken the assumptions on V, for instance, to relative boundedness.

As a corollary, we obtain a transport result for potentials that decay faster than short-range:

Corollary B.5. *Let $V \in C^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be such that*

$$\|(1+R)Q\chi_{|x|>R}V\|_\infty \in L^1([0,\infty), dr)$$
$$\|(1+R)\chi_{|x|>R}\nabla V\|_\infty \in L^1([0,\infty), dr).$$

Recall the sets

$$\mathcal{D}_a = \{\psi \in \mathcal{S}(\mathbb{R}^d) \mid \operatorname{supp}\hat{\psi} \Subset B_a^c\}$$
$$\mathcal{D} = \bigcup_{a>0}\mathcal{D}_a.$$

Then for all $\psi \in \mathcal{D}$, $\Omega\psi$ exists and exhibits ballistic transport in all directions.

Proof. This proof is very similar to the proof of Proposition 3.3, only allowing for short-range decay instead of compact support.

We note that we can write

$$\begin{aligned}\|Ve^{-itH_0}\psi\|_{H_1} &\le C(\|Ve^{-itH_0}\psi\| + \|\nabla Ve^{-itH_0}\psi\| + \|Ve^{-itH_0}\nabla\psi\|)\\ &\le C(\|Ve^{-itH_0}\psi\| + \|\nabla Ve^{-itH_0}\psi\| + \|Ve^{-itH_0}P\psi\|).\end{aligned}$$

We start with the first term and consider $0 < \varepsilon < a$

$$\begin{aligned}\|Ve^{-itH}\psi\| &\le \|V\chi_{|x|<\varepsilon t}e^{-itH_0}\psi\| + \|V\chi_{|x|>\varepsilon t}e^{-itH_0}\psi\|\\ &\le \|V\|_\infty\|\chi_{|x|<\varepsilon t}e^{-itH_0}\psi\| + \|V\chi_{|x|>\varepsilon t}\|_\infty\|\psi\|.\end{aligned}$$

By Proposition 3.1, since $\psi \in \mathcal{D}_a$, for any large enough $\ell > 0$, and some $C > 0$ we have the bound

$$\|\chi_{|x|<\varepsilon t}e^{-itH_0}\psi\|^2 \le \int\limits_{B_{\varepsilon t}} C(1+|x|+|t|)^{-\ell}\,dx \le C\frac{(\varepsilon t)^d}{(1+|t|)^\ell} < C(1+|t|)^{-\ell+d}.$$

Thus, we get that

$$\|Ve^{-itH}\psi\| \le C(1+|t|)^{-\ell} + \|V\chi_{|x|>\varepsilon t}\|_\infty\|\psi\|.$$

Noting that $P\psi \in \mathcal{D}_a$ as well, we can get the following

$$\|Ve^{-itH_0}\psi\|_{H_1} \le C[(1+|t|)^{-\ell} + \|V\chi_{|x|>\varepsilon t}\|_\infty\|\psi\| + \|V\chi_{|x|>\varepsilon t}\|_\infty\|P\psi\| + \|\nabla V\chi_{|x|>\varepsilon t}\|_\infty\|\psi\|],$$

so in particular we have that

$$\begin{aligned}&\int\limits_0^\infty (1+t)\|Ve^{-itH_0}\psi\|_{H^1}\,dt\\ &< C[\int\limits_0^\infty (1+t)(1+|t|)^{-\ell}\,dt + \|\psi\|_{H^1}\int\limits_0^\infty (1+t)\|V\chi_{|x|>\varepsilon t}\|_\infty\,dt + \|\psi\|\int\limits_0^\infty (1+t)\|\nabla V\chi_{|x|>\varepsilon t}\|_\infty\,dt].\end{aligned}$$

By assumption, we get that the last terms are finite, and the first is finite for $\ell > 0$ large enough. This shows that

$$\int_0^\infty (1+t)\|Ve^{-itH_0}P\psi\|\,dt < \infty$$

as well.

Finally, it remains to show that

$$\int_0^\infty (1+t)\|QVe^{-itH_0}\psi\|\,dt < \infty.$$

We note that for $\psi \in \mathcal{D}_a$ we can write

$$\begin{aligned}\|QVe^{-itH}\psi\| &\leq \|QV\chi_{|x|<\varepsilon t}e^{-itH_0}\psi\| + \|QV\chi_{|x|>\varepsilon t}e^{-itH_0}\psi\| \\ &\leq \varepsilon t\|V\|_\infty\|\chi_{|x|<\varepsilon t}e^{-itH_0}\psi\| + \|QV\chi_{|x|>\varepsilon t}\|_\infty\|\psi\|.\end{aligned}$$

By the above, we have that for some large enough $\ell > 0$, and some $C > 0$

$$\|\chi_{|x|<\varepsilon t}e^{-itH_0}\psi\| \leq C(1+|t|)^{-\ell},$$

so we find

$$\int_0^\infty (1+t)\|QVe^{-itH_0}\psi\|\,dt < \int_0^\infty (1+t)C(1+|t|)^{-\ell}\,dt + C\int_0^\infty (1+t)\|QV\chi_{|x|>\varepsilon t}\|_\infty\|\psi\|\,dt.$$

The second term is finite by assumption, and the first converges for ℓ large enough. □

Appendix C. A Form of Weierstrass Preparation

Recall that a Weierstrass polynomial on $U \subset \mathbb{R}^2$, a neighborhood of $(0,0)$, is a function of the form

$$F(x,y) = x^n + g_{n-1}(y)x^{n-1} + \cdots + g_0(y)$$

where each g_k is analytic and satisfies $g_k(0) = 0$.

Throughout this section, we will use the following notation. For a function g, we denote by Z_g its zero set. If f is a monic polynomial with complex roots $\{\alpha_i\}$, we recall that its discriminant is (up to a sign)

$$\Delta = \prod_{j\neq k}^n (\alpha_j - \alpha_k).$$

The complex analog of the following result may be gleaned from Chapter 1 of [5] (see also Chapter 6 of [30]). However, we include a proof below for the convenience of the reader.

Lemma C.1. *Let V be an open subset of $\mathbb{R}^2$ containing $(0,0)$. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is analytic, vanishes at $(0,0)$, and $z \mapsto f(0,z)$ is not identically 0. Then there exist open intervals I and J containing 0 with $I \times J \subset U$ and a Weierstrass polynomial F such that $Z_F = Z_f$ on $I \times J$ and the discriminant of F is not identically 0 on I.*

Proof. Let $\tilde{f}$ be the complexification of f, i.e. its extension to a complex analytic function on $\tilde{V}$ a neighborhood of $(0,0)$ in $\mathbb{C}^2$, which contains V. It suffices to find a Weierstrass polynomial F on some polydisc $D_1 \times D_2 \subset \mathbb{C}^2$ centered at $(0,0)$ with $Z_F = Z_{\tilde{f}}$ with a discriminant that does not vanish identically in $D_1 \times D_2$, because by the identity principle for single variable analytic functions, the discriminant cannot vanish on a real interval $I \subset D_1$ without vanishing on $D_1 \times D_2$.

With this in mind, by the assumption $f(0,z) \not\equiv 0$ and the identity principle, we may choose a disc centered at 0, $D_2 \subset \mathbb{C}$, so that 0 is the only zero of $f(0,z)$ inside D_2. By Rouché's theorem and continuity of $f(\cdot, z)$, we may find a disc $D_1 \subset \mathbb{C}$ containing 0 and so that for each $w \in D_1$, the function $f_w(z) : D_2 \to \mathbb{C}$ given by $f_w(z) = f(w,z)$ has exactly m zeroes (counted with multiplicity) in $D_2' = \frac{1}{2}D_2$. Now let $r := \sup_{w \in D_1} |\{z \in D_2 : f_w(z) = 0\}| \leq m$ be the maximal number of *geometrically distinct* zeroes of $f_w(z)$ for all $w \in D_1$. Let U be the set of $w \in D_1$ so that $z \mapsto f(w,z)$ has r geometrically distinct zeroes.

For any $w_0 \in U$, let $\alpha_1(w_0), \dots, \alpha_r(w_0)$ be the roots of $f_{w_0}(z)$ labeled arbitrarily. For each $1 \leq j \leq r$, let $C_j \subset D_2$ be a circle containing only $\alpha_j(w_0)$ so that $f_{w_0}(z)$ is non-zero on each C_j. By continuity and the compactness of the C_j, we may find a neighborhood of w_0, N, so that $f_w(z)$ is non-zero on each C_j for all $w \in N$. The argument principle shows that for all $w \in N$, the number of zeroes (counted with multiplicity) of $f_w(z)$ in each C_j is r_j, the multiplicity of $\alpha_j(w_0)$ as a zero of $f_{w_0}(z)$. However, by the maximality of r, there can only be exactly one geometrically distinct zero, which is given by the formula

$$\frac{1}{2\pi i r_j} \int\limits_{C_j} z \frac{f_w'(z)}{f_w(z)}\, dz$$

as a standard consequence of the residue theorem. Since this expression is clearly analytic in w (for instance, by differentiating with $\frac{\partial}{\partial \bar{w}}$ under the integral), we see that each α_j extends to an analytic function in some neighborhood of w, and in particular that U is open.

Now, let $\Delta(w) : U \to \mathbb{C}$ be given by

$$\Delta(w) = \Pi_{j \neq k}(\alpha_j(w) - \alpha_k(w)),$$

which is well-defined and analytic because the expression is independent of the labeling of the α_j. We wish to show that by defining $\Delta(w)$ to be 0 on $D_1 \setminus U$ we obtain a continuous function or equivalently that if $w_n \to w \in D_1 \setminus U$ then $\Delta(w_n) \to 0$. Thus, as roots approach the boundary of U in D_1, they must join, as the number of roots is smaller outside of U. To be precise, for each n let $\alpha_1(w_n), \dots, \alpha_r(w_n)$ be the r distinct roots of $f_{w_n}(z)$ and observe that the sequence $\alpha_1(w_n)$ lies in the compact set $\overline{D_2'}$ and therefore a subsequence $\alpha_1(w_{n_k})$ converges to some $\alpha_1(w)$. Applying this argument iteratively r-times, beginning with the sequence w_{n_k}, we find a subsequence of $\{w_n\}_{n=1}^{\infty}$, call it $\{w_{n_\ell}\}_{\ell=1}^{\infty}$, so that $\alpha_j(w_{n_\ell}) \to \alpha_j(w)$. By the continuity of $f(z,w)$, each $\alpha_j(w)$ is a root of f_w but because $w \notin U$ we must have that $\alpha_j(w) = \alpha_k(w)$ for some $j \neq k$. Examining the definition of $\Delta(w)$, we have shown that every subsequence $w_n \to w \in D_1 \setminus U$ has a further subsequence along which $\Delta(w_{n_\ell}) \to 0$, which establishes the desired continuity.

We have seen that we may extend Δ to all of D_1 as a function which is continuous on all of D_1 and analytic on U. It now follows from Radó's theorem [5, Section A1.5] that Δ is in fact analytic on D_1 so that its zero set (i.e. $D_1 \setminus U$) consists of isolated points. Now observe that on $U \times D_2$

$$F(w,z) = (z - \alpha_1(w)) \cdots (z - \alpha_r(w))$$

for each z is well-defined and analytic because it is independent of the labeling of the roots. Moreover, it is bounded because each $\alpha_j(w)$ lies in D_2' so that for each z we may extend it to an analytic function on D_1 by Riemann's removable singularity theorem. Thus, F is a Weierstrass polynomial whose roots coincide with those of f on U and therefore also on all of D_1 by the identity principle. By construction, its roots are simple outside of Z_Δ so its discriminant is non-vanishing there. □

Appendix D. Glossary

- We will denote by $\mathcal{H}$ the underlying Hilbert space: either $\ell^2(\mathbb{Z}^{n+m})$ or $L^2(\mathbb{R}^{n+m})$, depending on context.
- We will usually denote $d = n + m$ the total dimension of the space.
- We will denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space.

- The symbols $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ will also be used for the norm and inner product on $\mathbb{R}^{n+m}, \mathbb{Z}^{n+m}$, or $\mathcal{H}$. On the latter, it will be conjugate linear in the first coordinate, and linear in the second.
- We will use Q to denote the position operator:

$$Q\psi(x,y) = \vec{q}\psi(x,y)$$

  for $\vec{q} = (x_1, \dots, x_n, y_1, \dots, y_m)$.
- We will use X, Y to denote the position operator in the directions where the potential decays and where the potential is periodic, respectively. Their respective domains will be $D(X)$ and $D(Y)$.
- We will denote by P^x, P^y the momentum operator corresponding to the x, y-direction resp. we will denote by P the momentum operator.
- We will denote $A \Subset B$ when A is compactly supported inside of B.
- For an operator A we will denote by $A(t)$ the Heisenberg-evolved operator $A_H(t) = e^{itH}Ae^{-itH}$.
- We will denote χ_{vt} the indicator of the set $\{|x| > vt\}$, where the dimension is implicit. Similarly, $\chi_{vt}^c := \mathrm{Id} - \chi_{vt}$ will denote the indicator of the complement of the same set.
- $\mathbb{T}^*$ will denote the dual torus.
- The potential will have support in the x-direction in a ball of size R, and the periodicity in the y-direction will be of $L_1, \dots, L_m$.
- $W = \mathbb{R}^n \times \prod_{j=1}^m [0, L_j)$ is the fundamental cell.
- $\tilde{H}^2$ is the Sobolev space of periodic function on W.
- Uf will denote the partial Floquet transform of f.
- We use the following convention for the Fourier transform of $f \in L^2(\mathbb{R}^{n+m})$:

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = (2\pi)^{-\frac{n+m}{2}} \int_{\mathbb{R}^d} f(x)e^{-ix\xi}\,dx$$
$$\mathcal{F}^{-1}(\hat{f})(x) = (2\pi)^{-\frac{n+m}{2}} \int_{\mathbb{R}^d} \hat{f}(\xi)e^{ix\xi}\,d\xi.$$

- We use the following convention for the Fourier transform of $f \in \ell^2(\mathbb{Z}^{n+m})$:

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = (2\pi)^{-\frac{n+m}{2}} \sum_{n\in\mathbb{Z}^d} f(n)e^{-in\xi}$$
$$\mathcal{F}^{-1}(\hat{f})(n) = (2\pi)^{-\frac{n+m}{2}} \int_{\mathbb{T}} \hat{f}(\xi)e^{in\xi}\,d\xi.$$

- We will also use the following notation:

$$\Omega(t) = e^{itH}e^{-itH_0}$$
$$\Omega^*(t) = e^{itH_0}e^{-itH}$$

  and

$$\Omega = \Omega^- = \operatorname*{s-lim}_{t\to+\infty} e^{itH}e^{-itH_0}$$

  with domain $\mathcal{H}$.
- We will often use the shorthand $\frac{2\pi}{L}j$ to denote the vector in $\mathbb{R}^m$ with entries $\frac{2\pi}{L_k}j_k$.

- For a self-adjoint operator A and a Borel set $S \subset \mathbb{R}$, we write the spectral projection of A onto S as $\chi_S(A)$. We will denote by

$$\mathring{\mathcal{H}}_{\text{sur}} = \int_{\mathbb{T}^*}^{\oplus} \chi_{(-\infty,\|k\|^2)}(H(k))\, dk$$

  the subspace of unembedded surface states.
- $N(k)$ will be the number of eigenvalues of $H(k)$ below $\|k\|^2$, which may be 0 or infinity.
- $\mathring{\pi}_n(k)$ will be the eigenprojector associated to the nth eigenvalue of $H(k)$ below $\|k\|^2$ if $n \leq N(k)$ and 0 otherwise.
- We denote $S_n = \{k \in \mathbb{T}^* \mid \mathring{\pi}_n(k) \neq 0\}$.
- We will denote

$$\mathring{\mathcal{C}} := \bigcup_{M=1}^{\infty} \{\psi \in \mathring{\mathcal{H}}_{\text{sur}} \mid U\psi = \sum_{n=1}^{M} \mathring{\pi}_n(k) U\psi \text{ and for all } n \leq M,\, \mathring{\pi}_n(k)U\psi \in C^\infty(S_n)\},$$

  the dense subset on which directional ballistic transport will be proven.

The following notations are relevant to the discrete setting:

- We will denote the "twisted Laplacian" by Δ^k, the linear map on $\mathbb{C}^{\mathbb{Z}_L}$ given by

$$(\Delta^k \psi)(j) = e^{ik}\psi(j+1) + e^{-ik}\psi(j-1).$$

- v_n will denote the eigenvectors of the twisted Laplacian Δ^k.
- For energy E and quasimomentum k we will denote $e_j = E - 2\cos(k + \frac{2\pi j}{L})$.
- Define for $z \in \mathbb{C}$ the Joukowsky map $J(z) = z + \frac{1}{z}$. Then we let $\mu^\pm$ be the two analytic branches of the inverse, defined on the following domains $\mu^- : \mathbb{C} \setminus [-2,2] \to \mathbb{C} \setminus \overline{\mathbb{D}}$ and $\mu^+ : \mathbb{C} \setminus [-2,2] \to \mathbb{D}$ where $\mathbb{D}$ is the unit disc.
- For $\psi_x \oplus \psi_{x-1} = \Psi_x \in \mathbb{C}^L \oplus \mathbb{C}^L$, where

$$T_0(E,k)\Psi_x = \Big((E - \Delta^k)\psi_x - \psi_{x-1}\Big) \oplus \psi_x,$$

  or as a block matrix:

$$T_0(E,k) = \begin{pmatrix} E - \Delta^k & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}.$$

- We denote by

$$\mathcal{A} = \bigcup_{j \in \mathbb{Z}_L} \{(E,k) \in \mathbb{R}^2 \mid e_j(E,k) = 2\} \cup \{(E,k) \in \mathbb{R}^2 \mid e_j(E,k) = -2\},$$

  and denote $\mathcal{U} = \mathbb{R} \times (-\frac{\pi}{L}, \frac{\pi}{L}) \setminus \mathcal{A}$.
- We denote by $\mathcal{V}^\pm(E,k) \subset \mathbb{C}^L \oplus \mathbb{C}^L$ the subspace of vectors that decay at $\pm\infty$ under the action of T_0. These subspaces are also given by

$$\mathcal{V}^\pm(E,k) = \bigoplus_{\{j \in \mathbb{Z}_L \mid e_j(E,k) \notin [-2,2]\}} V_j^\pm(E,k).$$

- For $(E,k) \in \mathbb{R}^2$ let $\mathcal{I}^\pm(E,k) : \mathcal{V}^\pm \hookrightarrow \mathbb{C}^L \oplus \mathbb{C}^L$ be the inclusion of $\mathcal{V}^\pm$ in the full space and $\mathcal{P}^\pm(E,k) : \mathbb{C}^L \oplus \mathbb{C}^L \to \mathcal{V}^\pm \subset \mathbb{C}^L \oplus \mathbb{C}^L$ be the corresponding orthogonal projection.

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